Volumes of Revolution, Surface Area & Arc length

Exercises (with Sol'ns)(17 pages; 27/3/26)

(1) The region between the line $y = 6 - 2x$ and the curve $y = \frac{4}{x}$ is rotated about the y-axis through 360°. Find the exact volume generated.

Solution

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Solution

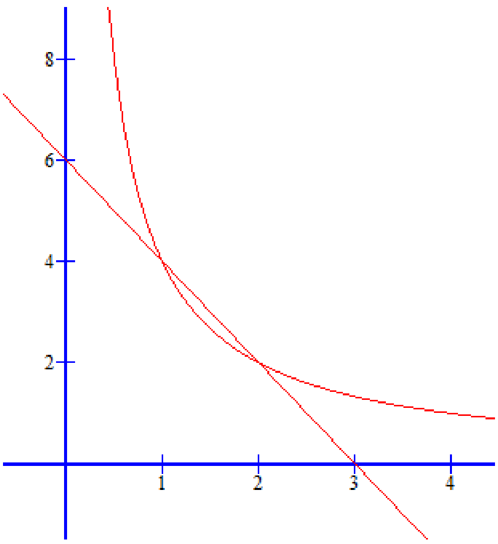
To find the points of intersection of the line and curve:

$\boldsymbol{y = 6 - 2x}$ and $\boldsymbol{y = \frac{4}{x} \Rightarrow \frac{4}{x} = 6 - 2x}$,

so that $\boldsymbol{4 = 6x - 2x^2}$, or $\boldsymbol{x^2 - 3x + 2 = 0}$

$\Rightarrow \boldsymbol{(x - 1)(x - 2) = 0}$,

so that the points of intersection are $\boldsymbol{(1, 4)}$ and $\boldsymbol{(2, 2)}$.



For the line, $\boldsymbol{x = 3 - \frac{y}{2}}$, and for the curve, $\boldsymbol{x = \frac{4}{y}}$

The required volume is $\boldsymbol{\int_2^4 \pi \left\{ \left(3 - \frac{y}{2}\right)^2 - \left(\frac{4}{y}\right)^2 \right\} dy}$

$\boldsymbol{= \pi \int_2^4 9 - 3y + \frac{y^2}{4} - \frac{16}{y^2} \; dy}$

$$= \pi[9y - \frac{3y^2}{2} + \frac{y^3}{12} + \frac{16}{y}]_2^4$$

$$= \pi\{\left(36 - 24 + \frac{16}{3} + 4\right) - \left(18 - 6 + \frac{2}{3} + 8\right)\}$$

$$= \pi\left\{-4 + \frac{14}{3}\right\} = \frac{2\pi}{3}\ units^3$$

(2) The region between the parabola $y^2 = 4x$, the x-axis and the line $x = 1$ is rotated about the x-axis through 360°.

(i) Find the exact volume generated:

(a) by integrating with respect to x

(b) by integrating with respect to the parameter t, where $x = t^2$ and $y = 2t$

(ii) Use the mean value of the function to carry out a rough check on your answer in (i).

(iii) Find the curved surface area associated with the volume generated in (i):

(a) by integrating with respect to x

(b) by integrating with respect to y

(c) by integrating with respect to t

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Solution

(i)(a) Volume = $\boldsymbol{\pi \int_0^1 y^2 dx = \pi \int_0^1 4x\, dx = \pi[2x^2]_0^1}$

$= \boldsymbol{2\pi(1-0) = 2\pi}$

(b) $\boldsymbol{x = 0 \Rightarrow t = 0; x = 1 \Rightarrow t = 1}$

Volume = $\boldsymbol{\pi \int_0^1 y^2 \frac{dx}{dt} dt = \pi \int_0^1 (2t)^2(2t) dt = 8\pi \int_0^1 t^3 dt}$

$= \boldsymbol{8\pi \left[\frac{1}{4}t^4\right]_0^1 = 2\pi(1-0) = 2\pi}$

(ii) Mean value = $\boldsymbol{\frac{1}{1-0}\int_0^1 \sqrt{4x}\, dx = 2\int_0^1 x^{\frac{1}{2}}\, dx}$

$= \boldsymbol{2\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 = \frac{4}{3}}$

Approximate volume is that of a cylinder of radius $\frac{4}{3}$ and length 1;

ie $\boldsymbol{\pi}\left(\frac{4}{3}\right)^{2}(\mathbf{1})=\frac{16}{9}\boldsymbol{\pi}$, which is reasonably close to $\mathbf{2}\boldsymbol{\pi}$.

(iii)(a) Curved SA = $\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y=\sqrt{4x}$ and $\frac{dy}{dx}=2(\frac{1}{2})x^{-\frac{1}{2}}$

so that $SA=4\pi\int_0^1 x^{\frac{1}{2}}\sqrt{1+\frac{1}{x}}\,dx=4\pi\int_0^1\sqrt{x+1}\,dx$

$=4\pi\left[\frac{(x+1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1=\frac{8\pi}{3}(2\sqrt{2}-1)$

(b) Curved SA = $\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$=\int_{x=0}^{x=1}2\pi y\sqrt{dx^2+dy^2}$ (informally)

$=\int_0^2 2\pi y\sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy$

$y^2=4x$, so that $\frac{dx}{dy}=\frac{1}{4}(2y)$

and SA = $2\pi\int_0^2 y\sqrt{\frac{y^2}{4}+1}\,dy$

Then, as $\frac{d}{dy}\left(\frac{y^2}{4}\right)=\frac{2y}{4}=\frac{y}{2}$, SA = $4\pi\left[\frac{\left(\frac{y^2}{4}+1\right)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^2$

$=\frac{8\pi}{3}(2\sqrt{2}-1)$

(c) Curved SA = $\int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$x = t^2$ and $y = 2t$, so that $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2$

and SA = $\int_0^1 2\pi(2t)\sqrt{4t^2 + 4}\, dt$

$= 4\pi \int_0^1 2t\sqrt{t^2 + 1}\, dt$

$= 4\pi\left[\frac{(t^2+1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1$

$= \frac{8\pi}{3}(2\sqrt{2} - 1)$

(3) The curve C has equation $y = \frac{1}{3}x^3 + \frac{1}{4x}$. The points A and B on C have x coordinates 1 and 2, respectively. Find the length of the arc from A to B.

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Solution

Length of arc $= \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

and $\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$

Note that $2x^2\left(-\frac{1}{4x^2}\right) = -\frac{1}{2}$

so that $1 + 2x^2\left(-\frac{1}{4x^2}\right) = \frac{1}{2} = 2x^2\left(\frac{1}{4x^2}\right)$

and $1 + \left(x^2 - \frac{1}{4x^2}\right)^2 = \left(x^2 + \frac{1}{4x^2}\right)^2$

So length of arc $= \int_1^2 x^2 + \frac{1}{4x^2}\, dx$

$= \left[\frac{1}{3}x^3 - \frac{1}{4x}\right]_1^2 = \left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{59}{24}$

(4) Use integration with respect to a suitable parameter to show that the surface area of a sphere of radius r is $4\pi r^2$.

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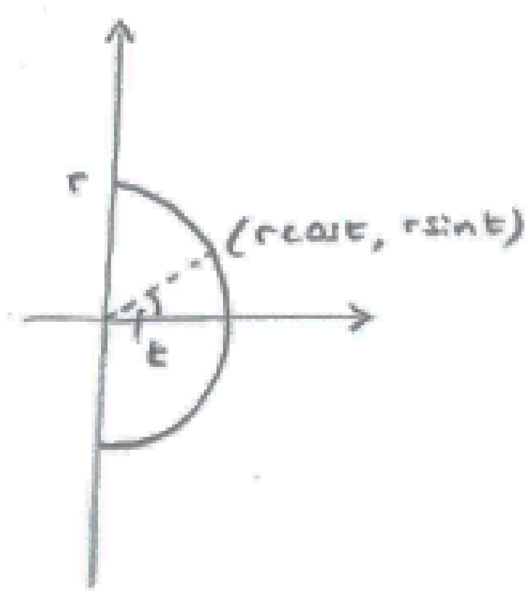
Solution

Consider the hemisphere obtained by rotating a quarter circle about the x-axis, as in the diagram below.



Surface area of hemisphere $= \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

$= \int_0^{\frac{\pi}{2}} 2\pi(rsint)\sqrt{(-rsint)^2 + (rcost)^2}\ \mathrm{dt}$

$= 2\pi r^2 \int_0^{\frac{\pi}{2}} sint\ dt$

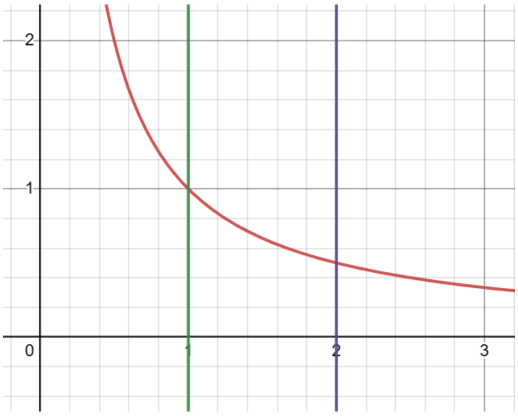
$= 2\pi r^2 [-cost]_0^{\frac{\pi}{2}}$

$= 2\pi r^2 (0 - (-1))$

$= 2\pi r^2$

Then the surface area of the sphere is double this, to give $4\pi r^2$.

(5) The region bounded by the curve $y = \frac{1}{x}$, the lines $x = 1, x = 2$, and the x-axis is rotated about the y-axis through $360°$. Find the volume generated.



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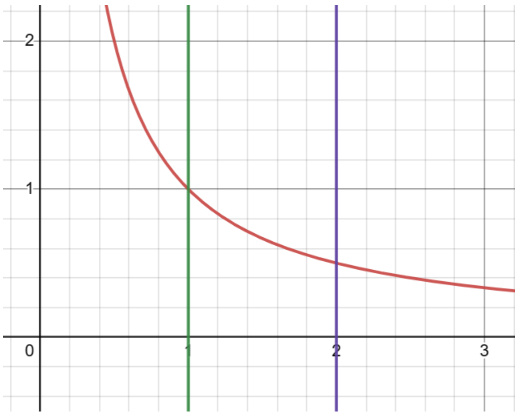

Solution

The region can be divided into two parts: the rectangle at the bottom, with base 1 and height $\frac{1}{2}$ (A), and the remainder (B).

Volume $= \pi(2^2 - 1^2)\left(\frac{1}{2}\right) + \pi \int_{\frac{1}{2}}^{1} x^2 - 1^2 dy$

$= \frac{3\pi}{2} + \pi \int_{\frac{1}{2}}^{1} y^{-2} - 1^2 dy$

$= \frac{3\pi}{2} + \pi[-y^{-1} - y]_{\frac{1}{2}}^{1}$

$= \frac{3\pi}{2} + \pi(-2 - \left[-\frac{5}{2}\right])$

$= 2\pi$

(6) The region between the curve $y = 4 - x^2$, the coordinate axes, and the line $x = 1$ is rotated about the y-axis. Find the volume of the solid generated.

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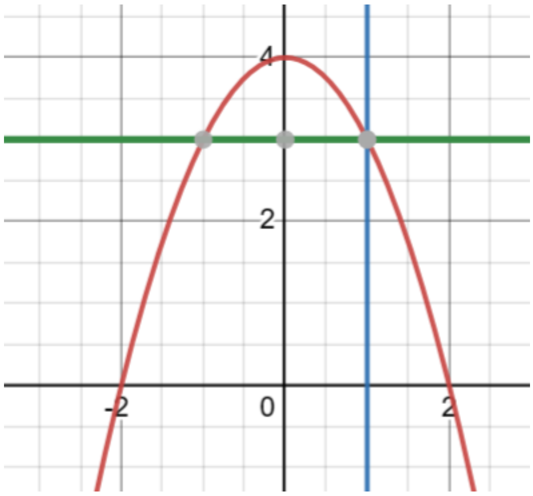
Solution



There are 2 parts to the volume:

$\pi \int_0^3 1^2 dy + \pi \int_3^4 (4 - y) dy$

$= \pi(3 - 0) + \pi[4y - \frac{1}{2}y^2]_3^4$

$= 3\pi + \pi[(16 - 8) - \left(12 - \frac{9}{2}\right)]$

$= \frac{7\pi}{2}\ units^3$

(7) Let V_x be the volume of revolution when the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated about the x-axis, and V_y the volume when it is rotated about the y-axis. Show that $\frac{V_x}{V_y} = \frac{b}{a}$.

(7) Let $\boldsymbol{V_x}$ be the volume of revolution when the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\mathbf{1}$ is rotated about the $\boldsymbol{x}$-axis, and $\boldsymbol{V_y}$ the volume when it is rotated about the $\boldsymbol{y}$-axis. Show that $\frac{V_x}{V_y}=\frac{b}{a}$.

Solution

$$V_x = 2\pi\int_0^a y^2 dx = 2\pi\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2\pi b^2\left[x-\frac{1}{3}\frac{x^3}{a^2}\right]_0^a$$

$$= 2\pi b^2\left(a-\frac{1}{3}a\right)$$

$$= \frac{4}{3}\pi b^2 a$$

By symmetry, $V_y = \frac{4}{3}\pi a^2 b$,

so that $\frac{V_x}{V_y}=\frac{b}{a}$