Vectors - Important Ideas (13 pages; 20/2/20)

[See notes on individual vector topics for more details.]

Contents

(1) Vector equation of a line

(i) Note the distinction between:

(a) the vector equation of the line passing through A and B (sometimes abbreviated to "... the line AB"; though strictly speaking it should read "... the line segment AB extended"), which is the position vector of a general point on the line, and

(b) the vector $\overrightarrow{AB}$, which is a vector in the direction of the line

(ii) If A and B are points on the line, and $\underline{d}$ is the direction of the line, then the following forms of the vector equation are possible:

$\underline{r} = \underline{a} + \lambda\underline{d}$ (where $\underline{a} = \overrightarrow{OA}$)

$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$

$\underline{r} = (1 - \lambda)\underline{a} + \lambda\underline{b}$

[this can be considered to be a weighted average of $\underline{a}$ and $\underline{b}$]

(iii) When asked for the vector equation of a line, it is essential to include the "$\underline{r} =$". Note that $\underline{r}$ can be replaced by $\begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, as appropriate, and that the vector equation can be written as 2 or 3 scalar equations.

(2) Cartesian form of a line in 3D

(a) The line $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ can be written as

$(\lambda =)\ \frac{x-2}{3} = \frac{y-4}{5} = \frac{z-6}{2}$

[More generally, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} a_1 + \lambda d_1 \\ a_2 + \lambda d_2 \\ a_3 + \lambda d_3 \end{pmatrix}$

becomes $\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$]

(b) The line $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$ would be written as

$\frac{x-2}{3} = \frac{y-4}{5}$, $z = 6$ (as $\frac{z-6}{0}$ is undefined)

It represents a line in the plane $z = 6$.

(c) The line $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ could be written as

$\lambda = \frac{x-2}{3}, y = 4 , z = 6$

It represents the line parallel to the x-axis passing through the point (0,4,6).

As x can take any value, the form $x = k, y = 4 , z = 6$ is preferable.

(3) Scalar product

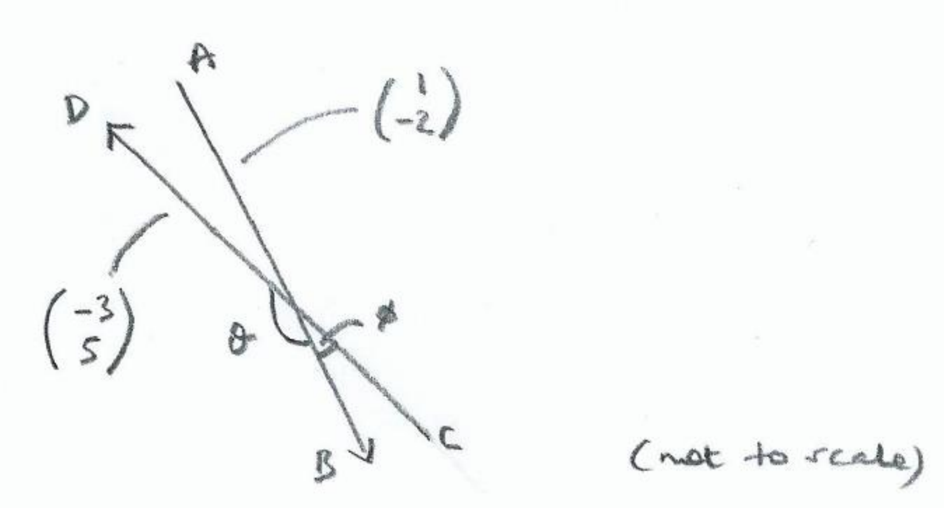
(i) $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \left(a_1\underline{i} + a_2\underline{j}\right) . \left(b_1\underline{i} + b_2\underline{j}\right)$

$= a_1b_1\underline{i}.\underline{i} + a_1b_2\underline{i}.\underline{j} + a_2b_1\underline{j}.\underline{i} + a_2b_2\underline{j}.\underline{j}$

$= a_1b_1 + 0 + 0 + a_2b_2$

$= a_1b_1 + a_2b_2$

(ii) Consider two line segments, $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$



[the locations of the lines are not important; only their directions]

Note that the gradients of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are $\frac{-2}{1} = -2$ and $\frac{5}{-3} = -\frac{5}{3}$.

We can find the angle θ between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ as follows:

$\overrightarrow{AB}.\overrightarrow{CD} = |\overrightarrow{AB}||\overrightarrow{CD}|cos\theta$,

giving $\begin{pmatrix} 1 \\ -2 \end{pmatrix}.\begin{pmatrix} -3 \\ 5 \end{pmatrix} = \sqrt{1^2 + (-2)^2}\sqrt{(-3)^2 + 5^2}\, cos\theta$

so that $cos\theta = \frac{-3-10}{\sqrt{5}\sqrt{34}} = \frac{-13}{\sqrt{170}}$

and hence $\theta = 175.601° = 175.6°$ (1dp)

Now consider the angle ϕ between $\overrightarrow{AB}$ and $\overrightarrow{DC} = -\begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

(note that the gradient of $\overrightarrow{DC}$ is still $-\frac{5}{3}$).

Then $\overrightarrow{AB}.\overrightarrow{DC} = |\overrightarrow{AB}||\overrightarrow{DC}|cos\phi = |\overrightarrow{AB}||\overrightarrow{CD}|cos\phi$, so that $\begin{pmatrix} 1 \\ -2 \end{pmatrix} . \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \sqrt{5}\sqrt{34}\, cos\phi$

and $cos\phi = \frac{3+10}{\sqrt{5}\sqrt{34}} = \frac{13}{\sqrt{170}}$

and hence $\phi = 180 - 175.6° = 4.4°$ (1dp)

This is consistent with the diagram above.

Note that, if asked to find the angle between the two lines, without any directions being specified (ie whether $\overrightarrow{AB}$ or $\overrightarrow{BA}$), it is customary to give the acute angle; ie 4.4° in this case.

(iii) $\underline{a} . \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = {a_1}^2 + {a_2}^2 + {a_3}^2 = |\underline{a}|^2$

(4) Equation of a plane

(a) $\underline{r} . \underline{n} = a . \underline{n} = d$ ('scalar product' form)

(normal $\underline{n}$, passing through the point with position vector $\underline{a}$)

(b) $\underline{r} . \hat{\underline{n}} = d'$, where $\hat{\underline{n}} = \frac{\underline{n}}{|\underline{n}|}$ and $d' = \frac{d}{|\underline{n}|}$ is the shortest distance from the plane to the Origin

(c) $n_1 x + n_2 y + n_3 z = d$ (cartesian form)

(derived from the 'scalar product' form)

(d) $\underline{r} = \underline{a} + \lambda\underline{d} + \mu\underline{e}$ (parametric form)

Notes for (d):

(i) This can be converted to the 'scalar product' form by taking

$\underline{n} = \underline{d} \times \underline{e}$

(Alternatively, obtain the cartesian form by eliminating λ and μ from $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} + \mu \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$)

(ii) $\underline{d}$ and $\underline{e}$ can be obtained from the points $\underline{a}$, $\underline{b}$ & $\underline{c}$ in the plane (eg $\underline{d} = \underline{b} - \underline{a}$ and $\underline{e} = \underline{c} - \underline{a}$)

(iii) To convert from cartesian to parametric form, let $x = s$ and $y = t$, to find z in terms of s and t, and giving

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ? \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ ? \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ ? \end{pmatrix}$$

(5) Angle between a line and a plane

To determine the angle between a line (with direction $\underline{d}$) and a plane (with normal $\underline{n}$): find the acute angle between $\underline{n}$ and $\underline{d}$, and subtract it from 90°.

(6) Angle between two planes

The angle between two planes is the acute angle between the normals of the planes.

(7) Vector perpendicular to a given (2D) vector

$\begin{pmatrix} -b \\ a \end{pmatrix}$ is perpendicular to $\begin{pmatrix} a \\ b \end{pmatrix}$

(8) Intersection of two lines

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$:

eliminate λ and μ from $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$

Note: If no solution exists (ie if the equations are not consistent), then the lines are skew.

(9) Intersection of a line and a plane

Point of intersection of the line $\underline{r} = \underline{a} + \lambda \underline{d}$ and the plane $\underline{r}.\underline{n} = b.\underline{n}$ $\Rightarrow (\underline{a} + \lambda \underline{d}).\underline{n} = b.\underline{n}$, giving a value for λ, and hence the required point on the line.

(10) Line of intersection of two planes

Method 1

Substitute $x = \lambda$ into the cartesian equations of the two planes, and find y and z in terms of λ, to give

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ ? \\ ? \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ ? \\ ? \end{pmatrix}$$

Method 2

Find two points, $\underline{a}$ and $\underline{b}$, that lie on both of the planes (and hence on the line); eg by setting $x = 0$ (for one point) and $y = 0$ (for another).

The equation of the intersecting line is then $\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$

Method 3

Find a point that lies on both of the planes; then for the direction of the line, take the vector product of the normals of the two planes (as the line will be perpendicular to both of these).

(11) Shortest distance from a point to a plane

To find the shortest distance from the point $\underline{p}$ to the plane

$\underline{r}.\underline{n} = d$:

Method 1

Obtain the unit normal vector $\underline{\hat{n}} = \frac{n}{|\underline{n}|}$

and rewrite $\underline{r}.\underline{n} = d$ as $\underline{r}.\underline{\hat{n}} = d'$, where $d' = \frac{d}{|\underline{n}|}$

Then consider the line $\underline{r} = \underline{p} + \lambda\underline{\hat{n}}$, and the point where this meets the plane; ie where $(\underline{p} + \lambda\underline{\hat{n}}).\underline{\hat{n}} = d'$

The value of λ obtained from this eq'n gives the required distance: $|\lambda|$.

Method 2

Create the equation of the plane passing through $\underline{p}$, parallel to the plane $\underline{r}.\underline{\hat{n}} = d'$, to give $\underline{r}.\underline{\hat{n}} = e'$

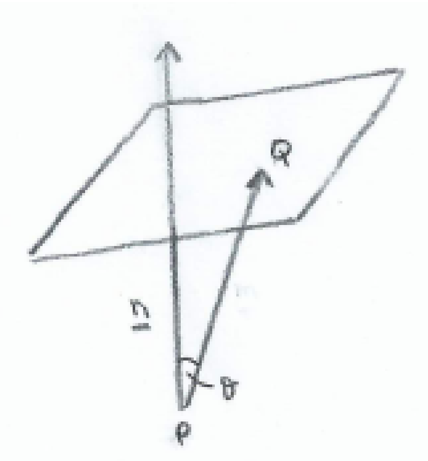
Then the required distance is $|d' - e'|$

Note: This is how the standard formula $\frac{|n_1p_1+n_2p_2+n_3p_3-d|}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$ is derived:

$$d' = \frac{d}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}} \quad \text{and} \quad e' = \underline{p}.\underline{\hat{n}} = \frac{n_1p_1+n_2p_2+n_3p_3}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$$

Method 3

Find any point Q in the plane (eg by setting $x = y = 0$ in the cartesian form).



The required distance will then be the projection of $\overrightarrow{PQ}$ onto $\underline{n}$ (the normal to the plane); namely $\frac{|\overrightarrow{PQ}.\underline{n}|}{|\underline{n}|}$

(12) Distance between two parallel planes

As for the shortest distance from a point to a plane, if the two planes are written in the form $\underline{r}.\underline{\hat{n}} = d'$ and $\underline{r}.\underline{\hat{n}} = e'$

(where $\underline{\hat{n}} = \frac{\underline{n}}{|\underline{n}|}$ is the unit normal vector, and $d' = \frac{d}{|\underline{n}|}$ (and similarly for e')),

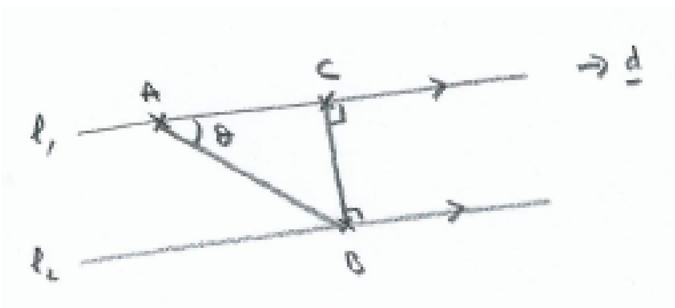
then the required distance is $|d' - e'|$

(13) Vector product

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & a_1 & b_1 \\ \underline{j} & a_2 & b_2 \\ \underline{k} & a_3 & b_3 \end{vmatrix} \text{ (or the transpose of this)}$$

(14) Shortest distance from a point to a line / distance between parallel lines

If A and B are given points on the two lines, and $\underline{d}$ is the common direction vector:



Method 1

Let C be the point on l_1 with parameter k, so that $\underline{c} = \underline{a} + k\underline{d}$ (*)

Then we require $\underline{d}.\left(\underline{c} - \underline{b}\right) = 0$

Solving this equation for k and substituting for k in (*) gives $\underline{c}$, and the distance between the two lines is then $|\underline{c} - \underline{b}|$.

Method 2

Having obtained the general point, C on l_1 in Method 1, we can minimise the distance BC by finding the stationary point of BC^2 (ie where $\frac{d}{dk}\left(BC^2\right) = 0$)

Method 3

As $BC = ABsin\theta,\ BC = \frac{|\overrightarrow{AB}\times\underline{d}|}{|\underline{d}|}$

Method 4 (2D lines)

The line equivalent of the formula $\frac{|n_1b_1+n_2b_2+n_3b_3-d|}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$ for the shortest distance from a point to a plane (see above) gives $\frac{|n_1b_1+n_2b_2-d|}{\sqrt{{n_1}^2+{n_2}^2}}$ as the shortest distance from the point $B(b_1, b_2)$ to the line $n_1x + n_2y = d$

(15) Vector product form of a line

$(\underline{r} - \underline{a}) \times \underline{d} = \underline{0}$ or $\underline{r} \times \underline{d} = \underline{a} \times \underline{d}$

(16) Vector perpendicular to two vectors

To find a vector perpendicular to the (3D) vectors $\underline{a}$ and $\underline{b}$:

Method 1

$\underline{a} \times \underline{b}$

Method 2

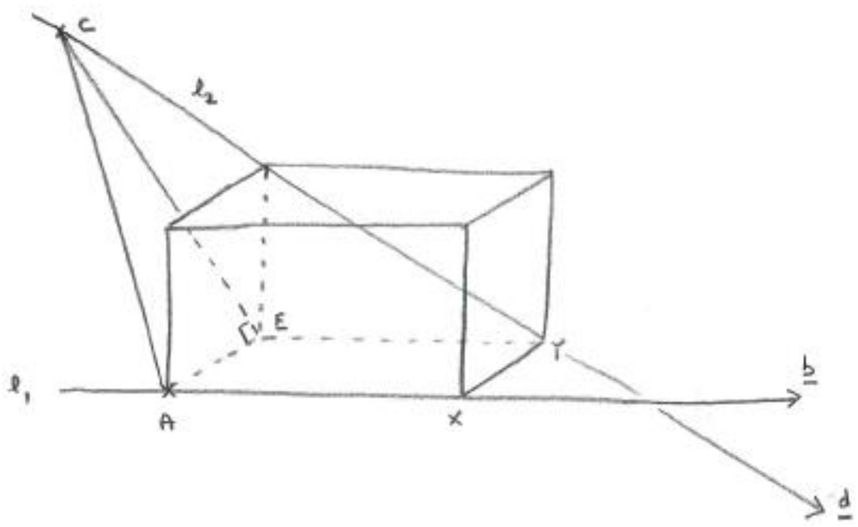
Let $\underline{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ be the required vector.

Then eliminate two of d_1, d_2 & d_3 from $\underline{d}.\underline{a} = 0$ and $\underline{d}.\underline{b} = 0$ (*)

to give a direction vector in terms of parameter $d_1, d_2\ or\ d_3$.

eg $\begin{pmatrix} d_1 \\ 2d_1 \\ 3d_1 \end{pmatrix}$, which is equivalent to the direction vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(17) Shortest distance between two skew lines



(l_1: $\underline{r} = \underline{a} + \lambda\underline{b}$ & l_2: $\underline{r} = \underline{c} + \mu\underline{d}$)

Method 1

$|(\underline{c} - \underline{a}).\frac{(\underline{b}\times\underline{d})}{|\underline{b}\times\underline{d}|}|$

Note: Two lines in 3D will intersect if $(\underline{c} - \underline{a}).(\underline{b} \times \underline{d}) = 0$

Method 2

Suppose that the shortest distance is XY, where X and Y are points on the two lines, with position vectors $\underline{r} = \underline{a} + \lambda_X\underline{b}$ & $\underline{r} = \underline{c} + \mu_Y\underline{d}$.

Then, if $\underline{n}$ is a vector normal to both $\underline{b}$ and $\underline{d}$ (eg $\underline{b} \times \underline{d}$)

$\underline{c} + \mu_Y\underline{d} = \underline{a} + \lambda_X\underline{b} + k\underline{n}$ (*)

(ie Y is reached by travelling first to X and then along XY) and XY will then $= k|\underline{n}|$

(*) gives 3 simultaneous equations in λ_X, μ_Y & k:

$$\begin{pmatrix} c_1 + \mu_Y d_1 \\ c_2 + \mu_Y d_2 \\ c_3 + \mu_Y d_3 \end{pmatrix} = \begin{pmatrix} a_1 + \lambda_X b_1 + kn_1 \\ a_2 + \lambda_X b_2 + kn_2 \\ a_3 + \lambda_X b_3 + kn_3 \end{pmatrix}$$, from which k can be found

Method 3

With X and Y defined as above, $\overrightarrow{XY} = (\underline{c} + \mu_Y \underline{d}) - (\underline{a} + \lambda_X \underline{b})$

and $\overrightarrow{XY}.\underline{b} = \overrightarrow{XY}.\underline{d} = 0$ (*)

Solving (*) enables λ_X & μ_Y to be determined,

from which $|\overrightarrow{XY}|$ can be found.