Vectors - Shortest distances (14 pages; 4/8/18)

(1) Shortest distance from a point to a plane

(See "Vectors - Planes" to convert between the scalar product and parametric forms of the equation of a plane, if necessary.)

Method 1

Example 1

Point, P is $\begin{pmatrix}1\\2\\3\end{pmatrix}$; plane has equation $4x+3y-12z=26$ (*)

The position vector of the point in the plane at the shortest distance from P is:

$\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\-12\end{pmatrix}$ for some λ (to be determined), as $\begin{pmatrix}4\\3\\-12\end{pmatrix}$ is the direction vector normal to the plane.

Since this point lies in the plane, it satisfies (*);

hence $4(1+4\lambda)+3(2+3\lambda)-12(3-12\lambda)=26$ (**)

giving $\lambda=\frac{4}{13}$

The shortest distance is the distance travelled from P to the plane, along the direction vector $\begin{pmatrix}4\\3\\-12\end{pmatrix}$; ie $\frac{4}{13}\left|\begin{pmatrix}4\\3\\-12\end{pmatrix}\right|=\frac{4}{13}(13)=4$

[Note: Had the direction vector been taken as $-\begin{pmatrix}4\\3\\-12\end{pmatrix}$, then we would have $\lambda=-\frac{4}{13}$ and the shortest distance would be given as

$|-\frac{4}{13}|\left|-\begin{pmatrix}4\\3\\-12\end{pmatrix}\right|$]

(Alternatively, the point in the plane can be written as

$\begin{pmatrix}1\\2\\3\end{pmatrix}+\frac{\mu}{13}\begin{pmatrix}4\\3\\-12\end{pmatrix}$, where $\frac{1}{13}\begin{pmatrix}4\\3\\-12\end{pmatrix}$ is a unit vector, and μ is found to be 4. Then the distance is just 4, as the point in the plane is 4 lots of the unit vector away from P.]

Example 2

In the special case where P is the origin O (and the plane has equation $4x + 3y - 12z = 26$ as before), (**) becomes

$4(4\lambda) + 3(3\lambda) - 12(-12\lambda) = 26$

ie $\lambda|\underline{n}|^2 = 26$, where $\underline{n}$ is the normal to the plane, $\begin{pmatrix}4\\3\\-12\end{pmatrix}$

As before, the shortest distance from O to the plane is

$\lambda|\underline{n}| = \frac{26}{|\underline{n}|} = \frac{26}{13} = 2$

Alternatively, if the equation of the plane is given in 'normalised' form (ie the direction vector has unit magnitude; the word 'normal' being used here in a different sense to that of the normal to a plane);

ie $\frac{4}{13}x + \frac{3}{13}y - \frac{12}{13}z = 2$, then the distance required is simply the right-hand side of the equation.

Method 2

Using the above example, we can find the equation of the plane parallel to $4x + 3y - 12z = 26$ and passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

The equation of the parallel plane will be

$4x + 3y - 12z = 4(1) + 3(2) - 12(3)$

ie $4x + 3y - 12z = -26$

From the special case of the Origin in Method 1, the distance between the two planes (and hence between the point and the plane) is $\frac{26-(-26)}{\sqrt{4^2+3^2+(-12)^2}} = \frac{52}{13} = 4$

Note: This method gives rise to the standard formula:

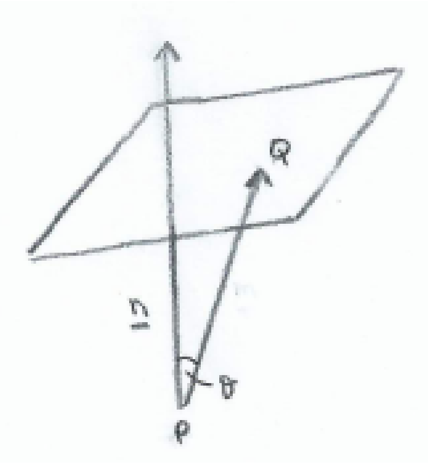
$\frac{|n_1p_1+n_2p_2+n_3p_3-d|}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$, as the shortest distance from the point $\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$

to the plane $n_1x + n_2y + n_3z = d$

[In this example, $|n_1p_1 + n_2p_2 + n_3p_3 - d| = |(-26) - 26|$

$= 26 - (-26)]$

Method 3a



Using the same example, where P is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and the plane has

equation $4x + 3y - 12z = 26$ (*),

we first of all find a point Q in the plane (as in the diagram above) and create the vector $\overrightarrow{PQ}$

The required distance will then be the projection of $\overrightarrow{PQ}$ onto $\underline{n}$ (the normal to the plane); namely $\frac{|\overrightarrow{PQ}.\underline{n}|}{|\underline{n}|}$

In this case, putting $y = z = 0$ (say) in (*) gives $x = \frac{13}{2}$, so that

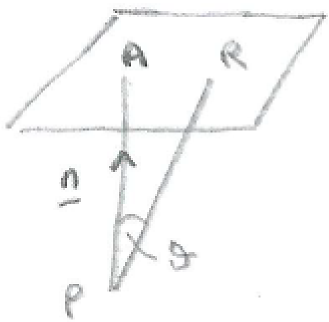
$Q = \begin{pmatrix} \frac{13}{2} \\ 0 \\ 0 \end{pmatrix}$, and $\overrightarrow{PQ} = \begin{pmatrix} \frac{11}{2} \\ -2 \\ -3 \end{pmatrix}$

Then $\overrightarrow{PQ}.\underline{n} = \left(\frac{11}{2}\right)(4) + (-2)(3) + (-3)(-12) = 52$

and the shortest distance $= \frac{52}{\sqrt{4^2+3^2+(-12)^2}} = \frac{52}{13} = 4$

Method 3b

With the same example, a variation on method 3a is to replace Q with a general point R, as in the diagram below.



Then $|PA| = \left|\overrightarrow{PR}.\widehat{\boldsymbol{n}}\right|$, where $\widehat{\boldsymbol{n}}$ is a unit normal to the plane,

ie $\widehat{\boldsymbol{n}} = \frac{1}{\sqrt{16+9+144}}\begin{pmatrix} 4 \\ 3 \\ -12 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 4 \\ 3 \\ -12 \end{pmatrix}$

(the direction could be reversed)

So $|PA| = \left|\left(\boldsymbol{r} - \begin{pmatrix}1\\2\\3\end{pmatrix}\right).\frac{1}{13}\begin{pmatrix}4\\3\\-12\end{pmatrix}\right|$

$= |\boldsymbol{r}.\frac{1}{13}\begin{pmatrix}4\\3\\-12\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix}.\frac{1}{13}\begin{pmatrix}4\\3\\-12\end{pmatrix}|$

$= |\frac{26}{13} - \frac{1}{13}(-26)|$

$= 4$

(2) Distance between two parallel planes

Using the method for finding the shortest distance from the Origin to a plane (Method 1, Example 2 of "Shortest distance from a point to a plane"), the two planes need first of all to be put into normalised form; the constant term of each equation then gives the distance of the plane from the origin, so that the distance between the planes is then the difference between the constant terms.

Example: Find the distance between the planes

$3x + 4y + 12z = 13$ and $3x + 4y + 12z = 39$

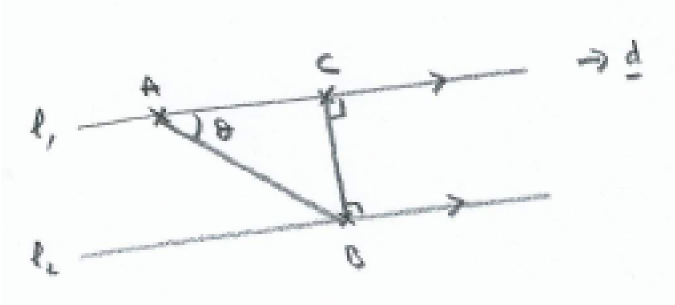
As $\sqrt{3^2 + 4^2 + 12^2} = 13$, the normalised equations are

$\frac{1}{13}(3x + 4y + 12z) = 1$ and $\frac{1}{13}(3x + 4y + 12z) = 3$

so that the distance between the planes is $3 - 1 = 2$

(3) Distance between parallel lines / shortest distance from a point to a line

Assuming that A and B are given points on the two lines, and that $\underline{d}$ is the common direction vector:



Method 1

Let C be the point on l_1 with parameter k, so that $\underline{c} = \underline{a} + k\underline{d}$ (*)

Then we require $\underline{d}.\left(\underline{c} - \underline{b}\right) = 0$

Solving this equation for k and substituting for k in (*) gives $\underline{c}$, and the distance between the two lines is then $|\underline{c} - \underline{b}|$.

Example

Let lines be $\underline{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ and $\underline{r} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

If $\underline{a} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$ and $\underline{c} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + k \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$,

then $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 + 3k + 2 \\ -k + 1 \\ 2 + k + 1 \end{pmatrix} = 0 \rightarrow 9 + 9k + k - 1 + k + 3 = 0$

$\rightarrow 11k + 11 = 0 \rightarrow k = -1$

Hence $\underline{c} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and the distance between the lines is

$\sqrt{(-2+2)^2 + (1+1)^2 + (1+1)^2} = \sqrt{8}$

Method 2

Having obtained the general point, $C = \begin{pmatrix} 1+3k \\ -k \\ 2+k \end{pmatrix}$ on l_1 in Method 1, we can minimise the distance BC by finding the stationary point of either BC or BC^2:

$BC^2 = (1+3k+2)^2 + (-k+1)^2 + (2+k+1)^2$

$= 11k^2 + 22k + 19$

Then $\frac{d}{dk}\,(BC^2) = 22k + 22$

and $\frac{d}{dk}\,(BC^2) = 0 \Rightarrow k = -1$, as before

Method 3

As $BC = ABsin\theta,\ BC = \frac{|\overrightarrow{AB} \times \underline{d}|}{|\underline{d}|}$

In the above example, $\overrightarrow{AB} = \begin{pmatrix} -3 \\ -1 \\ -3 \end{pmatrix}$ and $\overrightarrow{AB} \times \underline{d} =$

$\begin{vmatrix} \underline{i} & -3 & 3 \\ \underline{j} & -1 & -1 \\ \underline{k} & -3 & 1 \end{vmatrix} = -4\underline{i} - 6\underline{j} + 6\underline{k} = -2\begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$

Then $BC = \frac{2\sqrt{4+9+9}}{\sqrt{9+1+1}} = \frac{2\sqrt{22}}{\sqrt{11}} = \sqrt{8}$

Method 4

The line equivalent of the formula $\frac{|n_1b_1+n_2b_2+n_3b_3-d|}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$ for the shortest distance from a point to a plane (see above) gives

$\frac{|n_1b_1+n_2b_2-d|}{\sqrt{{n_1}^2+{n_2}^2}}$ as the shortest distance from the point $B(b_1,b_2)$ to the line $n_1x+n_2y=d$

Proof

The gradient of the line is $-\frac{n_1}{n_2}$, so that the unit direction vector of the line (in 3D) is $\underline{\hat{d}}=\frac{1}{\sqrt{{n_1}^2+{n_2}^2}}\begin{pmatrix}-n_2\\n_1\\0\end{pmatrix}$ $[\times\pm1]$

Let $A=\begin{pmatrix}a_x\\a_y\\0\end{pmatrix}$ be any point on the line, and let C be the point on the line nearest to the point B $\begin{pmatrix}b_1\\b_2\\0\end{pmatrix}$ (as in the initial diagram).

Then, by Method 3, BC $=|\overrightarrow{AB}\times\underline{\hat{d}}|$

$$=\frac{1}{\sqrt{{n_1}^2+{n_2}^2}}\left|\begin{pmatrix}b_1-a_x\\b_2-a_y\\0\end{pmatrix}\times\begin{pmatrix}-n_2\\n_1\\0\end{pmatrix}\right|$$

$$=\left|\frac{1}{\sqrt{{\mathrm{n}_1}^2+{\mathrm{n}_2}^2}}\begin{pmatrix}0\\0\\n_1(b_1-a_x)+n_2(b_2-a_y)\end{pmatrix}\right|$$

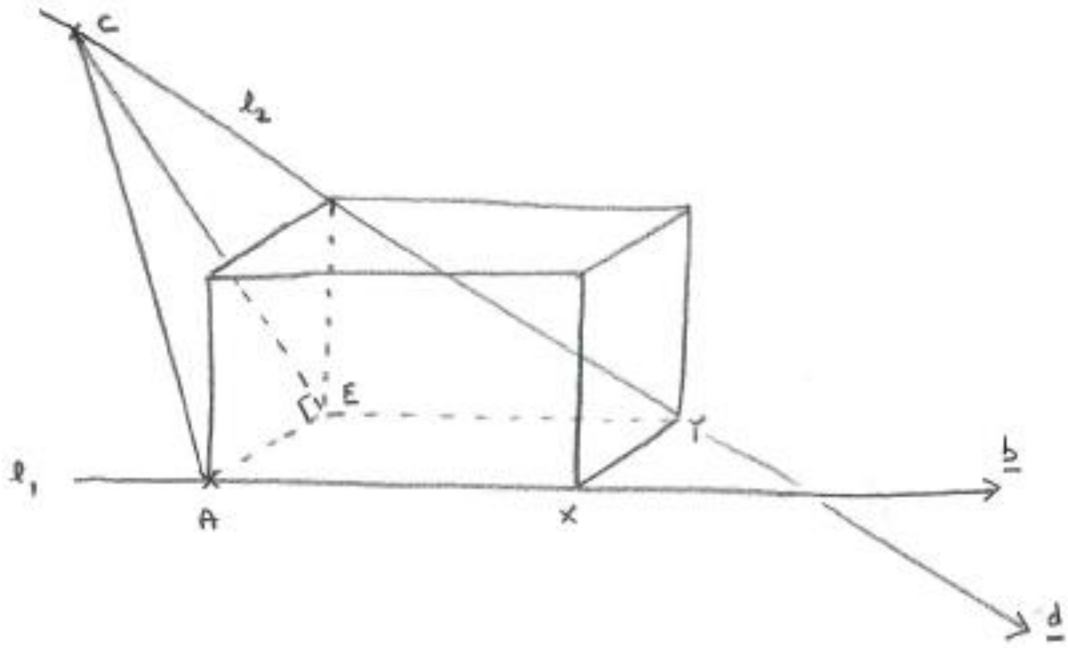
$$=\frac{n_1b_1+n_2b_2-d-(n_1a_x+n_2a_y-d)}{\sqrt{{\mathrm{n}_1}^2+{\mathrm{n}_2}^2}}$$

$=\frac{n_1b_1+n_2b_2-d}{\sqrt{{\mathrm{n}_1}^2+{\mathrm{n}_2}^2}}$, as required (as $\begin{pmatrix}a_x\\a_y\end{pmatrix}$ lies on $n_1x+n_2y=d$)

(4) Shortest distance between two skew lines

Method 1

eg l_1: $\underline{r} = \underline{a} + \lambda\underline{b}$ & l_2: $\underline{r} = \underline{c} + \mu\underline{d}$; A has position vector $\underline{a}$ etc



XY is shortest distance, as it is perpendicular to both l_1 and l_2

[To see that the above configuration is sufficiently general: given any two skew lines, we can start by identifying the shortest distance XY; this forces l_2 to be on the back face of the cuboid. l_2 can then have any direction in the plane of the back face, by changing the width and height of the cuboid.]

unit vector in direction of XY is $\frac{\underline{b}\times\underline{d}}{|\underline{b}\times\underline{d}|}$

AE = XY ; $\angle CEA = 90°$ (as CE is in the plane of the back face of the cuboid - because C lies on l_2)

So AE is the projection of $\underline{c} - \underline{a}$ onto the direction of XY; ie onto $\frac{\underline{b}\times\underline{d}}{|\underline{b}\times\underline{d}|}$

So XY = AE = $|(\underline{c} - \underline{a}).\frac{(\underline{b}\times\underline{d})}{|\underline{b}\times\underline{d}|}|$ (the modulus sign ensuring that the distance is +ve).

Note 1: This method can't be used to find the distance between two parallel lines, as $|\underline{b} \times \underline{d}| = 0$, since $sin\theta = 0$

Note 2: From the formula for XY, we can deduce that two lines in 3D will intersect if $(\underline{c} - \underline{a}).(\underline{b} \times \underline{d}) = 0$

Example 1a: To find the shortest distance between the lines

$l_1: \underline{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $l_2: \underline{r} = \begin{pmatrix} -1 \\ 12 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -14 \\ 4 \end{pmatrix}$

The direction normal to the two lines is

$\underline{b} \times \underline{d} = \begin{vmatrix} \underline{i} & 2 & 7 \\ \underline{j} & 2 & -14 \\ \underline{k} & -1 & 4 \end{vmatrix} = \begin{pmatrix} -6 \\ -15 \\ -42 \end{pmatrix}$; and we can take $\begin{pmatrix} 2 \\ 5 \\ 14 \end{pmatrix}$ instead

As $\sqrt{4 + 25 + 196} = 15$, the unit vector in this direction is $\frac{1}{15}\begin{pmatrix} 2 \\ 5 \\ 14 \end{pmatrix}$

We then require $\left(\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 12 \\ 5 \end{pmatrix}\right).\frac{1}{15}\begin{pmatrix} 2 \\ 5 \\ 14 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 1 \\ -14 \\ -5 \end{pmatrix}.\begin{pmatrix} 2 \\ 5 \\ 14 \end{pmatrix}$

$= \frac{1}{15}(2 - 70 - 70) = -\frac{138}{15} = -\frac{46}{5}$

so that the required distance is $\frac{46}{5}$ or 9.2

Method 2

Find the vector perpendicular to both $\underline{b}$ and $\underline{d}$, as in Method 1:
$\underline{n} = \underline{b} \times \underline{d}$

Then the equation of the plane with normal $\underline{n}$, containing line l_1 (ie the front face of the cuboid in Method 1) will be $\underline{r}.\underline{n} = \underline{a}.\underline{n}$

Similarly the equation of the plane with normal $\underline{n}$, containing line l_2 (ie the back face of the cuboid) will be $\underline{r}.\underline{n} = \underline{c}.\underline{n}$

The distance between these two planes (ie XY) is obtained by first adjusting the equations of the planes, so that they are based on a normal vector of unit magnitude.

Thus $\frac{\underline{r}.\underline{n}}{|\underline{n}|} = \frac{\underline{a}.\underline{n}}{|\underline{n}|}$ and $\frac{\underline{r}.\underline{n}}{|\underline{n}|} = \frac{\underline{c}.\underline{n}}{|\underline{n}|}$

Then $XY = \left|\frac{\underline{a}.\underline{n}}{|\underline{n}|} - \frac{\underline{c}.\underline{n}}{|\underline{n}|}\right|$ [See Vectors "Distance between two parallel planes" above.]

[Note that this method is algebraically equivalent to method 1.]

Example 1b (Lines as in 1a)

From Example 1a, $\frac{n}{|n|} = \frac{1}{15}\begin{pmatrix} 2 \\ 5 \\ 14 \end{pmatrix}$

Then the equations of the planes in which the front and back faces of the cuboid in method 1 lie are

$\frac{1}{15}(2x + 5y + 14z) = \frac{1}{15}[2(0) + 5(-2) + 14(0)] = -\frac{10}{15}$

and $\frac{1}{15}(2x + 5y + 14z) = \frac{1}{15}[2(-1) + 5(12) + 14(5)] = \frac{128}{15}$

So the distance between the two planes, and hence between the two lines is $= \frac{128-(-10)}{15} = \frac{138}{15} = \frac{46}{5} = 9.2$

Method 3

Referring to the earlier diagram, suppose that X and Y have position vectors $\underline{r} = \underline{a} + \lambda_X \underline{b}$ & $\underline{r} = \underline{c} + \mu_Y \underline{d}$ respectively.

Then, if $\underline{n}$ is a vector normal to both $\underline{b}$ and $\underline{d}$,

$\underline{c} + \mu_Y \underline{d} = \underline{a} + \lambda_X \underline{b} + k\underline{n}$ (*)

(ie Y is reached by travelling first to X and then along XY) and XY will then $= k|\underline{n}|$

(*) gives 3 simultaneous equations in λ, μ & k:

$\begin{pmatrix} c_1 + \mu_Y d_1 \\ c_2 + \mu_Y d_2 \\ c_3 + \mu_Y d_3 \end{pmatrix} = \begin{pmatrix} a_1 + \lambda_X b_1 + kn_1 \\ a_2 + \lambda_X b_2 + kn_2 \\ a_3 + \lambda_X b_3 + kn_3 \end{pmatrix}$, from which k can be found

Example 1c: (Lines as in 1a)

From Example 1a, $\underline{n} = \begin{pmatrix} 2 \\ 5 \\ 14 \end{pmatrix}$

We need to find k such that

$$\begin{pmatrix} -1 \\ 12 \\ 5 \end{pmatrix} + \mu_Y \begin{pmatrix} 7 \\ -14 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda_X \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + k \begin{pmatrix} 2 \\ 5 \\ 14 \end{pmatrix}$$

So $\quad 7\mu_Y - 2\lambda_X - 2k = 1$

$-14\mu_Y - 2\lambda_X - 5k = -14$

$$4\mu_Y + \lambda_X - 14k = -5$$

or $\begin{pmatrix} 7 & -2 & -2 \\ -14 & -2 & -5 \\ 4 & 1 & -14 \end{pmatrix}\begin{pmatrix} \mu_Y \\ \lambda_X \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ -14 \\ -5 \end{pmatrix}$

$$\rightarrow \begin{pmatrix} \mu_Y \\ \lambda_X \\ k \end{pmatrix} = \frac{1}{7(33) + 14(30) + 4(6)}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -6 & -15 & -42 \end{pmatrix}\begin{pmatrix} 1 \\ -14 \\ -5 \end{pmatrix}$$

$\rightarrow k = \frac{414}{675}$

and $XY = \frac{414}{675} \times \sqrt{4 + 25 + 196} = 9.2$

Method 4

As in method 3, suppose that X and Y have position vectors $\underline{r} = \underline{a} + \lambda_X \underline{b}$ & $\underline{r} = \underline{c} + \mu_Y \underline{d}$ respectively.

Then $\overrightarrow{XY} = \underline{c} + \mu_Y \underline{d} - (\underline{a} + \lambda_X \underline{b})$

and $\overrightarrow{XY}.\underline{b} = \overrightarrow{XY}.\underline{d} = 0$ (*)

Solving (*) enables λ_X & μ_Y to be determined,

from which $|\overrightarrow{XY}|$ can be found

Example 1d: (Lines as in 1a)

$$\overrightarrow{XY} = \begin{pmatrix} -1 \\ 12 \\ 5 \end{pmatrix} + \mu_Y \begin{pmatrix} 7 \\ -14 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} - \lambda_X \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 + 7\mu_Y - 2\lambda_X \\ 14 - 14\mu_Y - 2\lambda_X \\ 5 + 4\mu_Y + \lambda_X \end{pmatrix}$$

Then $\begin{pmatrix} -1 + 7\mu_Y - 2\lambda_X \\ 14 - 14\mu_Y - 2\lambda_X \\ 5 + 4\mu_Y + \lambda_X \end{pmatrix} . \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = 0$

and $\begin{pmatrix} -1 + 7\mu_Y - 2\lambda_X \\ 14 - 14\mu_Y - 2\lambda_X \\ 5 + 4\mu_Y + \lambda_X \end{pmatrix} . \begin{pmatrix} 7 \\ -14 \\ 4 \end{pmatrix} = 0$

$$\rightarrow -2 + 14\mu_Y - 4\lambda_X + 28 - 28\mu_Y - 4\lambda_X - 5 - 4\mu_Y - \lambda_X = 0$$

and

$-7 + 49\mu_Y - 14\lambda_X - 196 + 196\mu_Y + 28\lambda_X + 20 + 16\mu_Y + 4\lambda_X$

$= 0$

ie $\rightarrow 21 - 18\mu_Y - 9\lambda_X = 0$ or $7 - 6\mu_Y - 3\lambda_X = 0$

and $-183 + 261\mu_Y + 18\lambda_X = 0$ or $-61 + 87\mu_Y + 6\lambda_X = 0$

or $\begin{pmatrix} -6 & -3 \\ 87 & 6 \end{pmatrix} \begin{pmatrix} \mu_Y \\ \lambda_X \end{pmatrix} = \begin{pmatrix} -7 \\ 61 \end{pmatrix}$

$\rightarrow \begin{pmatrix} \mu_Y \\ \lambda_X \end{pmatrix} = \frac{1}{-36+261} \begin{pmatrix} 6 & 3 \\ -87 & -6 \end{pmatrix} \begin{pmatrix} -7 \\ 61 \end{pmatrix}$

$= \frac{1}{225} \begin{pmatrix} 141 \\ 243 \end{pmatrix} = \frac{1}{75} \begin{pmatrix} 47 \\ 81 \end{pmatrix}$

Then $\overrightarrow{XY} = \begin{pmatrix} -1 + 7\mu_Y - 2\lambda_X \\ 14 - 14\mu_Y - 2\lambda_X \\ 5 + 4\mu_Y + \lambda_X \end{pmatrix} = \frac{1}{75} \begin{pmatrix} 92 \\ 230 \\ 644 \end{pmatrix}$

and $|\overrightarrow{XY}| = \frac{1}{75}\sqrt{476100} = 9.2$