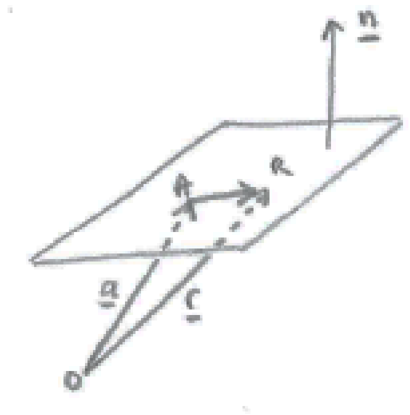
Vectors - Equation of plane (7 pages; 10/2/20)

(1) scalar product form

Let $\underline{a}$ be the position vector of a point in the plane,

and $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a general point in the plane.

Let $\underline{n}$ be a vector perpendicular to the plane (see below).



As $\underline{r} - \underline{a}$ and $\underline{n}$ are perpendicular, $(\underline{r} - \underline{a}).\underline{n} = 0$

$\Rightarrow \underline{r}.\underline{n} = \underline{a}.\underline{n} = p$ (a constant)

$\Rightarrow n_x x + n_y y + n_z z =$ p (Cartesian form)

Note: To find $\underline{n}$, given two direction vectors $\underline{d}_1$ and $\underline{d}_2$ in the plane: $\underline{n} = \underline{d}_1 \times \underline{d}_2$

Thus if $\underline{a}$, $\underline{b}$ & $\underline{c}$ are the position vectors of points in the plane, we can take $\underline{d}_1 = \underline{b} - \underline{a}$ and $\underline{d}_2 = \underline{c} - \underline{a}$, for example.

Example

If $\underline{a} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\underline{n} = \begin{pmatrix} -12 \\ 11 \\ -9 \end{pmatrix}$, then

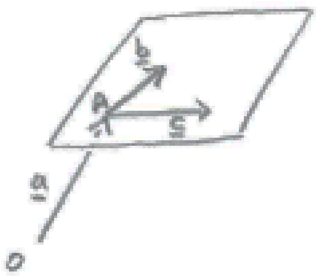
$-12x + 11y - 9z = -12(1) + 11(2) - 9(4) = -26$

(Another way of thinking of this is that, since $\underline{a}$ is a point in the plane, it is a solution of $\underline{r}.\underline{n} = p$, so that $p = \underline{a}.\underline{n}$, or $\underline{n}.\underline{a}$)

(2) Parametric form

This is an extension of the parametric form of the vector equation of a line.

Let $\underline{b}$ and $\underline{c}$ be non-zero vectors in the plane (that are not parallel to each other).



Then $\underline{r} = \underline{a} + (\lambda\underline{b} + \mu\underline{c})$

Note that $\underline{b}$ and $\underline{c}$ are direction vectors, whilst $\underline{a}$ is a position vector. $\underline{b}$ and $\underline{c}$ can of course be determined from 2 points $\underline{p}$ and $\underline{q}$ in the plane, as $\underline{p} - \underline{a}$ and $\underline{q} - \underline{a}$ (or $\underline{p} - \underline{q}$)

(3) Converting between cartesian and parametric forms

(a) to convert from cartesian to parametric form

Example

Suppose that the equation of the plane is $2x + 3y + z = 4$

Let $x = s$ and $y = t$, so that $z = 4 - 2s - 3t$ and a general point

is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ t \\ 4-2s-3t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$

(b) to convert from parametric to cartesian form

Example: $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + s\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

Method 1

Create the normal vector: $\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$= \begin{vmatrix} \underline{i} & -1 & 2 \\ \underline{j} & 3 & 3 \\ \underline{k} & 5 & 1 \end{vmatrix} = -12\underline{i} + 11\underline{j} - 9\underline{k}$

giving $-12x + 11y - 9z = -12(1) + 11(2) - 9(4) = -26$,

as the point $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ lies in the plane.

Method 2

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + s\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$\rightarrow \quad x = 1 - s + 2t$

$y = 2 + 3s + 3t$

$z = 4 + 5s + t$

Then eliminate s and t to obtain an equation in x, y & z.

(4) Obtaining the equation of a plane from 3 points in the plane

Example: Find the cartesian equation of the plane containing the points $(2, -1, 0), (1, 2, 1)$ and $(4, -3, -2)$

Solution

Method 1

Without loss of generality, let the equation of the plane be

$ax + by + cz = 1$

Then, substituting the 3 points into this equation:

$2a - b = 1 \quad (1)$

$a + 2b + c = 1 \quad (2)$

$4a - 3b - 2c = 1 \quad (3)$

Using (1) to eliminate b, (2) & (3) become:

$a + 2(2a - 1) + c = 1 \Rightarrow 5a + c = 3 \quad (2')$

$4a - 3(2a - 1) - 2c = 1 \Rightarrow -2a - 2c = -2 \Rightarrow a + c = 1 \quad (3')$

Then $(2') - (3')$ gives $4a = 2; a = \frac{1}{2}$, so that $c = \frac{1}{2}$ & $b = 0$

Thus the equation of the plane is $\frac{1}{2}x + \frac{1}{2}z = 1$, or $x + z = 2$.

Method 2

From the 3 points, 2 vectors in the plane are

$$\begin{pmatrix}1\\2\\1\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\3\\1\end{pmatrix} \text{ and } \begin{pmatrix}4\\-3\\-2\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}2\\-2\\-2\end{pmatrix},$$

so that the parametric form of the equation of the plane is

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\3\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\-2\end{pmatrix}$$

which can be written as

$x = 2 - \lambda + 2\mu$ (1)

$y = -1 + 3\lambda - 2\mu$ (2)

$z = \lambda - 2\mu$ (3)

Then we can eliminate λ & μ to obtain the cartesian equation:

From (1), $\lambda = 2 + 2\mu - x$

and then (2) $\Rightarrow y = -1 + 3(2 + 2\mu - x) - 2\mu$

$\Rightarrow y + 3x - 2 = 4\mu$ (4)

and (3) $\Rightarrow z = (2 + 2\mu - x) - 2\mu$

$\Rightarrow z + x = 2$ or $x + z = 2$.

Method 3

Without loss of generality, let the equation of the plane be

$x + by + cz = d$

2 vectors in the plane are

$$\begin{pmatrix}1\\2\\1\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}-1\\3\\1\end{pmatrix} \text{ and } \begin{pmatrix}4\\-3\\-2\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}2\\-2\\-2\end{pmatrix} \text{ or } \begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

The normal to the plane $\begin{pmatrix}1\\b\\c\end{pmatrix}$ must be perpendicular to the vectors in the plane, so that $\begin{pmatrix}1\\b\\c\end{pmatrix}.\begin{pmatrix}-1\\3\\1\end{pmatrix}=0$ and $\begin{pmatrix}1\\b\\c\end{pmatrix}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=0$,

and hence $-1+3b+c=0$ and $1-b-c=0$,

so that $3b+c=1$ and $b+c=1$,

giving $b=0$ and $c=1$

Then the equation of the plane is $x+z=\begin{pmatrix}1\\0\\1\end{pmatrix}.\begin{pmatrix}2\\-1\\0\end{pmatrix}$,

as $\begin{pmatrix}2\\-1\\0\end{pmatrix}$ is a point in the plane; ie $x+z=2$

Method 4

As Method 3, except that the normal is obtained from

$$\begin{pmatrix}-1\\3\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\begin{pmatrix}-2\\0\\-2\end{pmatrix} \text{ or } \begin{pmatrix}1\\0\\1\end{pmatrix}$$