Vectors: Miscellaneous - Exercises (with Sol'ns)

(6 pages; 7/5/26)

(1) Find c, a & b such that $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} = a\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$

[ie such that the 3 vectors are not linearly independent]

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Solution

As the position vector $\begin{pmatrix}2\\3\\c\end{pmatrix}$ is in the plane containing the Origin and the position vectors $\begin{pmatrix}-1\\0\\3\end{pmatrix}$ & $\begin{pmatrix}0\\2\\4\end{pmatrix}$, it follows that $\begin{pmatrix}2\\3\\c\end{pmatrix}$ is perpendicular to the normal to that plane; ie perpendicular to

$\begin{pmatrix}-1\\0\\3\end{pmatrix} \times \begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{vmatrix}\underline{i} & -1 & 0\\ \underline{j} & 0 & 2\\ \underline{k} & 3 & 4\end{vmatrix}$; so that

$\begin{pmatrix}2\\3\\c\end{pmatrix} . \begin{vmatrix}\underline{i} & -1 & 0\\ \underline{j} & 0 & 2\\ \underline{k} & 3 & 4\end{vmatrix} = 0$, and thus $\begin{vmatrix}2 & -1 & 0\\ 3 & 0 & 2\\ c & 3 & 4\end{vmatrix} = 0$

Alternative Approach 1

The 3 vectors form a parallelepiped of zero volume, so that the scalar triple product of the vectors is zero.]

Alternative Approach 2

The required relation can be written as

$\begin{pmatrix}2\\3\\c\end{pmatrix} - a\begin{pmatrix}-1\\0\\3\end{pmatrix} - b\begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, which implies a solution of $x\begin{pmatrix}2\\3\\c\end{pmatrix}+y\begin{pmatrix}-1\\0\\3\end{pmatrix}+z\begin{pmatrix}0\\2\\4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

other than $x=y=z=0$,

and for there to be more than one solution to this matrix equation, we require the determinant to be zero.

Then $\begin{vmatrix}2 & -1 & 0\\3 & 0 & 2\\c & 3 & 4\end{vmatrix}=0 \Rightarrow 2(-6)-(-1)(12-2c)=0$

$\Rightarrow c=0$

$\begin{pmatrix}2\\3\\0\end{pmatrix}=a\begin{pmatrix}-1\\0\\3\end{pmatrix}+b\begin{pmatrix}0\\2\\4\end{pmatrix}$

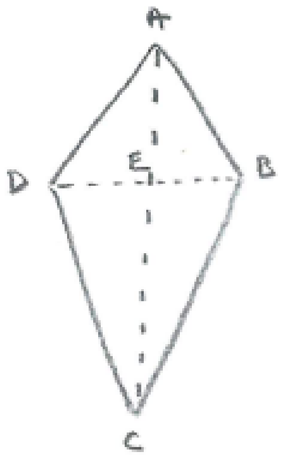
$\Rightarrow 2=-a$

$3=2b$

& $0=3a+4b$

so that $a=-2$ & $b=\frac{3}{2}$

(2) In the diagram below, ABCD is a kite. Find $\overrightarrow{OD}$ if $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 4/3 \\ 7 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 4/3 \\ 2 \end{pmatrix}$ & $\overrightarrow{OC} = \begin{pmatrix} 6 \\ 16/3 \\ 2 \end{pmatrix}$



[from AEA, June 2009]

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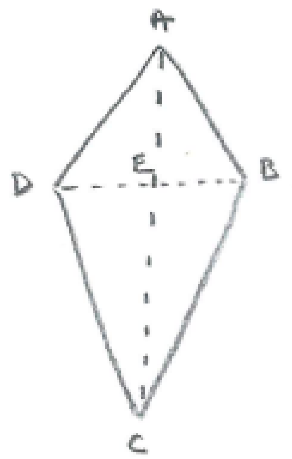

Solution

We need to take account of the special features of this case, namely that AC is perpendicular to BD (*) and bisects BD (**) (which enables D to be uniquely determined from A, B & C).

We can take an alternative route to D, in order to involve the other points, writing:

$\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BE}$ [this takes account of (**)]

To record the fact that E lies on AC, we write $\overrightarrow{BE} = \overrightarrow{BA} + \lambda\overrightarrow{AC}$

We also need to take account of (*): $\overrightarrow{BE}.\overrightarrow{AC} = 0$

Now $\overrightarrow{BA} = \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix}$, so that $\overrightarrow{BE} = \begin{pmatrix} -5+7\lambda \\ 4\lambda \\ 5-5\lambda \end{pmatrix}$

$\overrightarrow{BE}.\overrightarrow{AC} = 0$ gives 7(-5+7λ) + 4(4λ) -5(5-5λ) = 0 so that $90\lambda = 60$ and $\lambda = 2/3$

Hence $\overrightarrow{BE} = \begin{pmatrix} -1/3 \\ 8/3 \\ 5/3 \end{pmatrix}$

Then $\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BE} = \begin{pmatrix} 4 - 2/3 \\ \frac{4}{3} + 16/3 \\ 2 + 10/3 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 20/3 \\ 16/3 \end{pmatrix} = 2/3 \begin{pmatrix} 5 \\ 10 \\ 18 \end{pmatrix}$