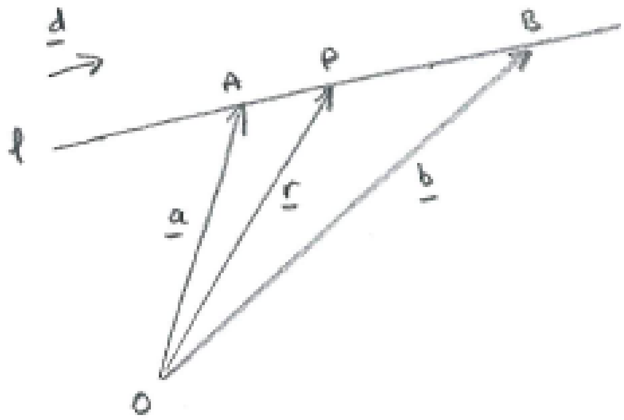
Vectors - Equation of line (4 pages; 4/8/18)

(1) Parametric form



The vector equation of the line l through the points A & B can be written in various (parametric) forms:

(a) $\underline{r} = \underline{a} + \lambda\underline{d}$

(b) $\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$

(c) $\underline{r} = (1 - \lambda)\underline{a} + \lambda\underline{b}$

(a weighted average of $\underline{a}$ & $\underline{b}$; when $\lambda = 0, \underline{r} = \underline{a}$; when $\lambda = 1$, $\underline{r} = \underline{b}$; when $\lambda = \frac{1}{2}, \underline{r}$ is the average of $\underline{a}$ & $\underline{b}$; the diagram shows $\lambda = \frac{1}{3}$)

(d) (in 2D case; similarly for 3D)

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ or $\begin{pmatrix} a_1 + \lambda d_1 \\ a_2 + \lambda d_2 \end{pmatrix}$

where $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\underline{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ is any vector in the direction from A to B

(normally d_1 & d_2 are chosen to be integers with no common factor)

(2) Note the difference between (a) the vector equation of the line through the points A & B and (b) the vector $\overrightarrow{AB}$: The vector $\overrightarrow{AB}$ has magnitude |AB| (the distance between A & B) and is in the direction from A to B.

Whereas the vector equation of the line through A & B is the position vector $\underline{r}$ of a general point P on the line, with completely different magnitude and direction to that of the vector $\overrightarrow{AB}$.

(3) Relation to the cartesian form

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \Rightarrow \lambda = \frac{x-a_1}{d_1} = \frac{y-a_2}{d_2}$

$\Rightarrow y = a_2 + \frac{d_2}{d_1}.(x - a_1)$

the straight line through (a_1, a_2) with gradient $\frac{d_2}{d_1}$

[In 3D: $\lambda = \frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$]

(4) 3D Example: line through (1, 0, 1) and (0, 1, 0)

$\underline{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

Hence $\underline{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} 1-\lambda \\ \lambda \\ 1-\lambda \end{pmatrix}$

and $\lambda = \frac{x-1}{-1} = \frac{y-0}{1} = \frac{z-1}{-1}$

(5) Direction Cosines

If the direction vector of a line is $\underline{d} = d_1\underline{i} + d_2\underline{j} + d_3\underline{k}$,

we can write $d_1 = |\underline{d}|cos\theta_1$, so that the **direction cosines** are

defined as $l_1(= cos\theta_1) = \frac{d_1}{|\underline{d}|}, l_2 = \frac{d_2}{|\underline{d}|}$ & $l_3 = \frac{d_3}{|\underline{d}|}$

and $\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$ is a unit vector

[Direction cosines are usually applied in the 3D case, where there isn't a gradient as such.]

Notes

(i) The letters l, m and n are often used instead of l_1, l_2 and l_3.

(ii) The **direction ratios** of a line are just d_1, d_2 and d_3 (or any 3 numbers in the same ratio).

(6) Vector product form (3D lines only)

$\underline{r} = \underline{a} + \lambda\underline{d}$ can be written as $(\underline{r} - \underline{a}) \times \underline{d} = \underline{0}$

(since $\underline{r} - \underline{a}$ and $\underline{d}$ are parallel)

or $\underline{r} \times \underline{d} = \underline{a} \times \underline{d}$

eg line through (1, 0, 1) and (0, 1, 0):

$$\underline{d} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

$$\underline{a} \times \underline{d} = \begin{vmatrix}\underline{i} & 1 & -1\\ \underline{j} & 0 & 1\\ \underline{k} & 1 & -1\end{vmatrix} = -i + k = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

Thus equation is $\underline{r} \times \begin{pmatrix}-1\\1\\-1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$

Note: Textbooks sometimes write the determinant with the elements transposed (it gives the same result though).

To reconcile $\underline{r} \times \begin{pmatrix}-1\\1\\-1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$ with $\underline{r} = \begin{pmatrix}1-\lambda\\\lambda\\1-\lambda\end{pmatrix}$:

LHS = $\begin{vmatrix}i & x & -1\\ j & y & 1\\ k & z & -1\end{vmatrix} = (-y-z)\underline{i} - (-x+z)\underline{j} + (x+y)\underline{k}$

Hence $\begin{pmatrix}-y-z\\x-z\\x+y\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$

Let $y = \lambda$; then $x = 1-\lambda$ and $z = 1-\lambda$