Vectors: Lines & Planes - Exercises (with Sol'ns)

(64 pages; 7/5/26)

(1) Given that the line $\underline{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ can also be written as $\begin{pmatrix} 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, find μ in terms of λ

(1) Given that the line $\underline{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ can also be written as

$\begin{pmatrix} 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, find μ in terms of λ

Solution

$\begin{pmatrix} 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} + (2 + \mu) \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Thus $2 + \mu = -\lambda$, and so $\mu = -\lambda - 2$

(2) Find a vector equation of the line that passes through the point (1,2) and is perpendicular to the line $\underline{r} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

(2) Find a vector equation of the line that passes through the point (1,2) and is perpendicular to the line $\underline{r} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

Solution

Method 1

The gradient of the given line is $\frac{-1}{4}$, so that the gradient of the perpendicular line is 4.

Then a vector equation of the required line is

$\underline{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Method 2 (much longer, but good practice!)

Let P be the intersection of the given line (L, say) and the perpendicular line through Q(1,2). Then P can be represented as $\begin{pmatrix} 3+4\lambda \\ 4-\lambda \end{pmatrix}$, for some λ to be determined.

Then, as L is perpendicular to QP, $\begin{pmatrix} 4 \\ -1 \end{pmatrix} . \begin{pmatrix} 3+4\lambda-1 \\ 4-\lambda-2 \end{pmatrix} = 0$

[noting that $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ is the direction vector of L; not to be confused with $\begin{pmatrix} 3+4\lambda \\ 4-\lambda \end{pmatrix}$, which the position vector of a point on L]

so that $4(2+4\lambda) - (2-\lambda) = 0$,

and hence $17\lambda + 6 = 0$, and $\lambda = -\frac{6}{17}$

Thus P is $\begin{pmatrix} 3+4\left(-\frac{6}{17}\right) \\ 4-\left(-\frac{6}{17}\right) \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 27 \\ 74 \end{pmatrix}$

And a vector equation of the line through P and Q is

$$\underline{r} = \begin{pmatrix}1\\2\end{pmatrix} + \lambda[\begin{pmatrix}1\\2\end{pmatrix} - \frac{1}{17}\begin{pmatrix}27\\74\end{pmatrix}]$$

or $\underline{r} = \begin{pmatrix}1\\2\end{pmatrix} + \frac{\lambda}{17}\begin{pmatrix}17-27\\34-74\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + \frac{\lambda}{17}\begin{pmatrix}-10\\-40\end{pmatrix}$

or $\underline{r} = \begin{pmatrix}1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\4\end{pmatrix}$

(3)(i) Show that the line $\underline{r} = \underline{a} + t\underline{b}$ and the plane $\underline{r}.\underline{n} = d$ intersect at the point $\underline{r} = \underline{a} + \left(\frac{d-\underline{a}.\underline{n}}{\underline{b}.\underline{n}}\right)\underline{b}$

(ii) Find the intersection of the line $\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and the plane $\underline{r}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -2$

(iii) Find the angle between the line and the plane in (ii).

(3)(i) Show that the line $\underline{r} = \underline{a} + t\underline{b}$ and the plane $\underline{r}.\underline{n} = d$ intersect at the point $\underline{r} = \underline{a} + \left(\frac{d-\underline{a}.\underline{n}}{\underline{b}.\underline{n}}\right)\underline{b}$

(ii) Find the intersection of the line $\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and the plane $\underline{r}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -2$

(iii) Find the angle between the line and the plane in (ii).

Solution

(i) $(\underline{a} + t\underline{b}).\underline{n} = d \Rightarrow \underline{a}.\underline{n} + t\underline{b}.\underline{n} = d$

$\Rightarrow t = \frac{d-\underline{a}.\underline{n}}{\underline{b}.\underline{n}}$

$\Rightarrow \underline{r} = \underline{a} + (\frac{d-\underline{a}.\underline{n}}{\underline{b}.\underline{n}})\underline{b}$

(ii) Applying the result in (i):

$\underline{a}.\underline{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1$

and $\underline{b}.\underline{n} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1$

so that $\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (\frac{-2-[-1]}{-1})\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \sqrt{1+4+0}\,.\sqrt{1+1+0}\,cos\theta$

$\Rightarrow cos\theta = \frac{1-2+0}{\sqrt{5}.\sqrt{2}} = -\frac{1}{\sqrt{10}}$

$\Rightarrow \theta = 108.43495°$

The required angle is:

$90 - (180 - 108.43495) = 18.43495 = 18.4°$ (1dp)

(4) Find the acute angle between the line $\frac{x-4}{-3} = \frac{y+2}{5}$, $z = -2$ and the plane $2x - z = 7$.

(4) Find the acute angle between the line $\frac{x-4}{-3} = \frac{y+2}{5}$, $z = -2$ and the plane $2x - z = 7$.

Solution

The angle between the direction vector of the line and the normal to the plane is given by

$$\begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix} . \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \sqrt{(-3)^2 + 5^2 + 0^2}\sqrt{2^2 + 0^2 + (-1)^2}\, cos\theta$$

$\Rightarrow -6 = \sqrt{34}\sqrt{5}\, cos\theta \Rightarrow cos\theta = -\frac{6}{\sqrt{170}} \Rightarrow \theta = 117.399 = 117.4°$ (1dp) [this is usually the preferred degree of accuracy for an angle given in degrees]

This means that the acute angle between these two directions is

$180 - 117.4 = 62.6$, and the angle between the plane itself and the line is therefore $90 - 62.6 = 27.4°$

(5)(i)(a) Find the acute angle between the line $\frac{x}{2} = \frac{y+1}{-3} = \frac{z-2}{1}$ and the plane $x + y - 2z = 5$

(b) Show that the same angle is obtained if the line is written in the form

$\frac{x}{-2} = \frac{y+1}{3} = \frac{z-2}{-1}$ (ie without rearranging into the form in (a))

(ii)(a) Find the acute angle between the planes $x + 4y - 3z = 7$

and $x - y + 4z = 2$

(b) Find the acute angle between the planes $x + 4y - 3z = 7$ and

$-x + y - 4z = 2$ (again, without rearranging the equation)

Solution

(i)(a) The angle between the line and the normal to the plane is given by

$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \sqrt{14}\sqrt{6}\, cos\theta$, so that $cos\theta = \frac{-3}{\sqrt{14}\sqrt{6}} = -0.32733$

and $\theta = 109.107°$

The acute angle between these vectors is then $180 - 109.107 = 70.893°$

The acute angle between the line and plane is then

$90 - 70.893 = 19.1°$ (1dp)

(b) $\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \sqrt{14}\sqrt{6}\, cos\theta \Rightarrow cos\theta = \frac{3}{\sqrt{14}\sqrt{6}} = 0.32733$

and $\theta = 70.893°$

As we have already found the acute angle between the line and the normal, the acute angle between the line and the plane is $90 - 70.893 = 19.1°$ (1dp)

(ii) The angle between the normals to the planes is given by

$\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \sqrt{26}\sqrt{18}\, cos\theta$, so that $cos\theta = \frac{-15}{\sqrt{26}\sqrt{18}} = -0.69338$

and $\theta = 133.898°$

The acute angle between the planes themselves is $\mathbf{180 - 133.898 = 46.1°}$

(ii)(b) The angle between the normals to the planes is given by

$\begin{pmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{-3} \end{pmatrix} . \begin{pmatrix} \mathbf{-1} \\ \mathbf{1} \\ \mathbf{-4} \end{pmatrix} = \sqrt{\mathbf{26}}\sqrt{\mathbf{18}}\ \boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{\mathbf{15}}{\sqrt{\mathbf{26}}\sqrt{\mathbf{18}}} = \mathbf{0.69338}$

and $\boldsymbol{\theta} = \mathbf{46.1°}$

The acute angle between the planes is also $\mathbf{46.1°}$.

(6) Find the line that is the reflection of the line $\frac{x-2}{3} = \frac{y}{4} = \frac{z+1}{1}$ in the plane $x - 2y + z = 4$

(6) Find the line that is the reflection of the line $\frac{x-2}{3}=\frac{y}{4}=\frac{z+1}{1}$ in the plane $x-2y+z=4$

Solution

Let the intersection of the line and the plane be P, and suppose that Q is some other point on the line. Then we can find the reflection of Q in the plane (Q' say), by dropping a perpendicular from Q onto the plane, and then the required line will pass through P and Q'.

Writing the equation of the line as $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\1\end{pmatrix}$ and substituting into the equation of the plane:

$$(2+3\lambda)-2(4\lambda)+(-1+\lambda)=4 \Rightarrow -4\lambda=3;\ \lambda=-\frac{3}{4}$$

so that P is $\begin{pmatrix}2\\0\\-1\end{pmatrix}-\frac{3}{4}\begin{pmatrix}3\\4\\1\end{pmatrix}=\begin{pmatrix}-\frac{1}{4}\\-3\\-\frac{7}{4}\end{pmatrix}$

Setting $\lambda=1$ (say), we can take Q to be $\begin{pmatrix}2\\0\\-1\end{pmatrix}+\begin{pmatrix}3\\4\\1\end{pmatrix}=\begin{pmatrix}5\\4\\0\end{pmatrix}$

Now consider the perpendicular line dropped from Q onto the plane. Its direction vector is that of the normal to the plane, and so it has equation

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

Let R be the point where the perpendicular line intersects the plane. Substituting into the equation of the plane gives:

$$(5+\lambda)-2(4-2\lambda)+(\lambda)=4 \Rightarrow 6\lambda=7;\ \lambda=\frac{7}{6}$$

So R is $\begin{pmatrix}5\\4\\0\end{pmatrix}+\frac{7}{6}\begin{pmatrix}1\\-2\\1\end{pmatrix}$, and Q' will be $\begin{pmatrix}5\\4\\0\end{pmatrix}+2\left(\frac{7}{6}\right)\begin{pmatrix}1\\-2\\1\end{pmatrix}=$
$\begin{pmatrix}\frac{22}{3}\\-\frac{2}{3}\\\frac{7}{3}\end{pmatrix}$

Then , as P is $\begin{pmatrix}-\frac{1}{4}\\-3\\-\frac{7}{4}\end{pmatrix}$, the equation of the reflected line will be:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-\frac{1}{4}\\-3\\-\frac{7}{4}\end{pmatrix}+\lambda\left[\begin{pmatrix}\frac{22}{3}\\-\frac{2}{3}\\\frac{7}{3}\end{pmatrix}-\begin{pmatrix}-\frac{1}{4}\\-3\\-\frac{7}{4}\end{pmatrix}\right]=\frac{1}{12}\begin{pmatrix}-3+\lambda(88+3)\\-36+\lambda(-8+36)\\-21+\lambda(28+21)\end{pmatrix}$$

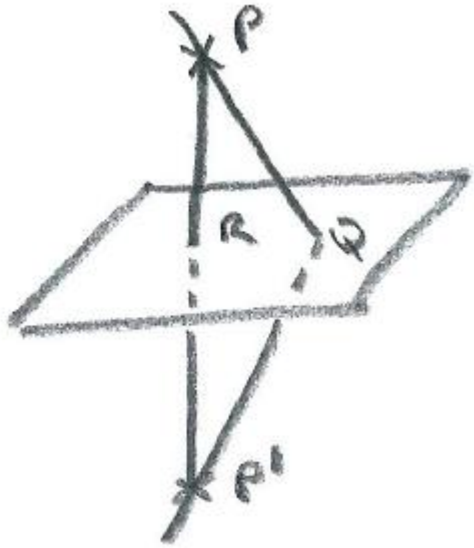
$$\frac{1}{12}\begin{pmatrix}-3+91\lambda\\-36+28\lambda\\-21+49\lambda\end{pmatrix}$$

or, in cartesian form: $\frac{x+\frac{3}{12}}{91}=\frac{y+\frac{36}{12}}{28}=\frac{z+\frac{21}{12}}{49}$ or $\frac{x+\frac{3}{12}}{13}=\frac{y+\frac{36}{12}}{4}=\frac{z+\frac{21}{12}}{7}$

(7) Find the reflection of the line $\frac{x-2}{3} = \frac{y+4}{1}; z = 3$ in the plane $y = 4$

(7) Find the reflection of the line $\frac{x-2}{3} = \frac{y+4}{1}; z = 3$ in the plane $y = 4$

Solution



Let P be $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$, say.

Q is intersection of the line and plane :

Line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

Substituting into the eq'n of the plane: $-4 + \lambda = 4 \Rightarrow \lambda = 8$

So Q is $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + 8 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 26 \\ 4 \\ 3 \end{pmatrix}$

Line PR is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

R is intersection of PR and the plane:

$-4 + \mu = 4 \Rightarrow \mu = 8$

So P' is $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + 2(8)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix}$

Eq'n of P'Q is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta[\begin{pmatrix} 26 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix}]$

ie $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta\begin{pmatrix} 24 \\ -8 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta'\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

or $\frac{x-2}{3} = \frac{y-12}{-1}; z = 3$

(8) Find the angle between the planes $x = 2$ and $y + 2z = 3$

(8) Find the angle between the planes $x = 2$ and $y + 2z = 3$

Solution

The two normal vectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$,

so that $cos\theta = 0$ (where θ is the angle between the normals), and hence the planes are perpendicular.

[The plane $x = 2$ is parallel to $x = 0$; ie the y-z plane. The plane

$y + 2z = 3$ can be formed from the line $y + 2z = 3$ in the y-z plane, extended in the positive and negative x directions (ie perpendicular to the y-z plane.]

(9) Find the cartesian form of the plane

$$\underline{r} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

(9) Find the cartesian form of the plane

$$\underline{r} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Solution

$$\underline{n} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{vmatrix} \underline{i} & 1 & 2 \\ \underline{j} & 4 & 3 \\ \underline{k} & 4 & 1 \end{vmatrix} = \begin{pmatrix} -8 \\ 7 \\ -5 \end{pmatrix}$$

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}\right) . \begin{pmatrix} -8 \\ 7 \\ -5 \end{pmatrix} = 0 \Rightarrow -8x + 7(y+2) - 5(z+1) = 0$$

$\Rightarrow -8x + 7y - 5z = -9$ or $8x - 7y + 5z = 9$

Alternative version (once $\underline{n}$ has been found)

Let plane be $-8x + 7y - 5z = p$

As $\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$ lies in the plane, $-8(0) + 7(-2) - 5(-1) = p$;

so $p = -9$ etc

Alternative method

Eliminate s & t from the 3 simultaneous equations.

(10) Find the plane containing the points $(2, -1, 4), (-3, 4, 2)$ and $(1, 0, 5)$, in Cartesian form

(10) Find the plane containing the points $(2,-1,4),(-3,4,2)$ and $(1,0,5)$, in Cartesian form

Solution

Method 1

In parametric form, it is:

$$\underline{r} = \begin{pmatrix}2\\-1\\4\end{pmatrix} + \lambda\left[\begin{pmatrix}-3\\4\\2\end{pmatrix} - \begin{pmatrix}2\\-1\\4\end{pmatrix}\right] + \mu[\begin{pmatrix}1\\0\\5\end{pmatrix} - \begin{pmatrix}2\\-1\\4\end{pmatrix}]$$

or $\underline{r} = \begin{pmatrix}2\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}-5\\5\\-2\end{pmatrix} + \mu\begin{pmatrix}-1\\1\\1\end{pmatrix}$

$x = 2 - 5\lambda - \mu \quad (1)$

$y = -1 + 5\lambda + \mu \quad (2)$

$z = 4 - 2\lambda + \mu \quad (3)$

Eliminating λ & μ: $(1) + (2) \Rightarrow x + y = 1$

Method 2

The normal to the plane is perpendicular to both

$\begin{pmatrix}-5\\5\\-2\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\1\end{pmatrix}$; eg $\begin{pmatrix}-5\\5\\-2\end{pmatrix} \times \begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{vmatrix}\underline{i} & -5 & -1\\ \underline{j} & 5 & 1\\ \underline{k} & -2 & 1\end{vmatrix}$

$= \begin{pmatrix}7\\7\\0\end{pmatrix}$ or $\begin{pmatrix}1\\1\\0\end{pmatrix}$, so that eq'n is $\underline{r}.\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\4\end{pmatrix}.\begin{pmatrix}1\\1\\0\end{pmatrix}$ or $x+y=1$

(11) (i) Find a vector that is perpendicular to both

$\begin{pmatrix} 7 \\ 0 \\ -10 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

(ii) Use (i) to find the plane that passes through the points with position vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 8 \\ 2 \\ -7 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$

(11) (i) Find a vector that is perpendicular to both
$\begin{pmatrix} 7 \\ 0 \\ -10 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

(ii) Use (i) to find the plane that passes through the points with

position vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 8 \\ 2 \\ -7 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$

Solution

(i) **Method 1**

$$\begin{pmatrix} 7 \\ 0 \\ -10 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{vmatrix} \underline{i} & 7 & 1 \\ \underline{j} & 0 & 3 \\ \underline{k} & -10 & -1 \end{vmatrix} = \begin{pmatrix} 30 \\ -3 \\ 21 \end{pmatrix}$$

[From the theory of the vector product,] this is perpendicular to the given vectors (as is $\frac{1}{3}\begin{pmatrix} 30 \\ -3 \\ 21 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 7 \end{pmatrix}$).

Method 2

Let the required vector be $\begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$

Then $\begin{pmatrix} 7 \\ 0 \\ -10 \end{pmatrix} . \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 0$, so that $7 - 10b = 0$ & $b = \frac{7}{10}$

And $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 0$, so that $1 + 3a - b = 0$,

and $a = \frac{1}{3}\left(\frac{7}{10} - 1\right) = -\frac{1}{10}$

Thus (multiplying by 10) a suitable vector is $\begin{pmatrix} 10 \\ -1 \\ 7 \end{pmatrix}$.

(ii) Let $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 8 \\ 2 \\ -7 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$ represent the points A, B & C, respectively.

Then $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 0 \\ -10 \end{pmatrix}$ & $\overrightarrow{AC} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

From (i), a vector that is perpendicular to $\overrightarrow{AB}$ & $\overrightarrow{AC}$ (and therefore a normal to the plane) is $\begin{pmatrix} 10 \\ -1 \\ 7 \end{pmatrix}$.

So the equation of the plane is

$$\underline{r} \, . \begin{pmatrix} 10 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} . \begin{pmatrix} 10 \\ -1 \\ 7 \end{pmatrix} = 10 - 2 + 21 = 29$$

or (in cartesian form) $10x - y + 7z = 29$ [as $\underline{r} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$]

(12) (i) Find the plane containing the points
$(3,0,-1),(5,2,-3)$ and $(4,2,4)$, in parametric form

(ii) Hence find the equation of the plane in Cartesian form.

(12) (i) Find the plane containing the points $(3,0,-1), (5,2,-3)$ and $(4,2,4)$, in parametric form

(ii) Hence find the equation of the plane in Cartesian form.

Solution

(i) A general point on the plane is

$$\underline{r} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + \lambda[\begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}] + \mu[\begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}]$$

ie $\underline{r} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

(ii) We can write

$x = 3 + 2\lambda + \mu$

$y = 2\lambda + 2\mu$

$z = -1 - 2\lambda + 5\mu$

Using the eq'n for x to eliminate μ:

$y = 2\lambda + 2(x - 3 - 2\lambda)$ or $y = -2\lambda + 2x - 6$

$z = -1 - 2\lambda + 5(x - 3 - 2\lambda)$ or $z = -12\lambda + 5x - 16$

Then, eliminating λ:

$-12\lambda = 6y - 12x + 36$ & also $z - 5x + 16$,

so that $6y - 12x + 36 = z - 5x + 16$

Hence $-7x + 6y - z = -20$ or $7x - 6y + z = 20$

[To check that this plane contains the given points:

$7(3) - 6(0) + (-1) = 20,$

$7(5) - 6(2) + (-3) = 20$

& $7(4) - 6(2) + 4 = 20$]

(13) Write the following Cartesian equations of planes in parametric form:

(i) $x + y + z = 1$ (ii) $x + y = 1$ (iii) $x = 1$

(13) Write the following Cartesian equations of planes in parametric form:

(i) $x+y+z=1$ (ii) $x+y=1$ (iii) $x=1$

Solution

(i) For this case, we can set $x=\lambda$ and $y=\mu$, if a parametric form of this type is possible:

$$\underline{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\a\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\b\end{pmatrix}+\mu\begin{pmatrix}0\\1\\c\end{pmatrix}\quad(*)$$

This means that a will be 1.

Then $\begin{pmatrix}1\\0\\b\end{pmatrix}$ & $\begin{pmatrix}0\\1\\c\end{pmatrix}$ will be valid direction vectors in the plane,

provided that they are perpendicular to the normal to the

plane; ie that there are solutions to $\begin{pmatrix}1\\0\\b\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix}=0$ and

$\begin{pmatrix}0\\1\\c\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix}=0$; so $b=c=-1$

Thus a possible parametric form is

$$\underline{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}+\mu\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

[Alternatively, $x+y+z=1\Rightarrow\lambda+\mu+z=1$,

so that $z=1-\lambda-\mu$, and so $a=1, b=-1$ & $c=-1$]

(ii) In this case, $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ b \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ c \end{pmatrix}$

isn't possible, as $x = y = 0$ doesn't satisfy $x + y = 1$

Instead, we can try $x = \lambda$ and $z = \mu$, so that $y = 1 - \lambda$

This gives $\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

and, as a check, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ satisfies $x + y = 1$,

and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$ & $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$

(iii) Here, we can obviously set $x = 1, y = \lambda, z = \mu$

Thus a possible parametric form is

$\underline{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(14) Find the shortest distance between the lines

$\frac{\boldsymbol{x-2}}{\boldsymbol{4}} = \frac{\boldsymbol{y-1}}{\boldsymbol{3}} = \frac{\boldsymbol{z+3}}{\boldsymbol{2}}$ and $\frac{\boldsymbol{x+5}}{\boldsymbol{7}} = \frac{\boldsymbol{y}}{\boldsymbol{1}} = \frac{\boldsymbol{z-1}}{\boldsymbol{3}}$

(14) Find the shortest distance between the lines

$\frac{x-2}{4}=\frac{y-1}{3}=\frac{z+3}{2}$ and $\frac{x+5}{7}=\frac{y}{1}=\frac{z-1}{3}$

Solution

[Note: The following method is probably the quickest way of finding the shortest distance, but doesn't generate the points on the lines that are closest.]

Shortest distance, $\boldsymbol{D}=\left|\frac{(\underline{\boldsymbol{d}}_1\times\underline{\boldsymbol{d}}_2).(\underline{\boldsymbol{a}}_1-\underline{\boldsymbol{a}}_2)}{|\underline{\boldsymbol{d}}_1\times\underline{\boldsymbol{d}}_2|}\right|$

where $\underline{\boldsymbol{d}}_1=\begin{pmatrix}4\\3\\2\end{pmatrix}$, $\underline{\boldsymbol{d}}_2=\begin{pmatrix}7\\1\\3\end{pmatrix}$, $\underline{\boldsymbol{a}}_1=\begin{pmatrix}2\\1\\-3\end{pmatrix}$, $\underline{\boldsymbol{a}}_2=\begin{pmatrix}-5\\0\\1\end{pmatrix}$

Then $\underline{\boldsymbol{d}}_1\times\underline{\boldsymbol{d}}_2=\begin{vmatrix}\underline{\boldsymbol{i}} & 4 & 7\\ \underline{\boldsymbol{j}} & 3 & 1\\ \underline{\boldsymbol{k}} & 2 & 3\end{vmatrix}=\begin{pmatrix}7\\2\\-17\end{pmatrix}$ and $\underline{\boldsymbol{a}}_1-\underline{\boldsymbol{a}}_2=\begin{pmatrix}7\\1\\-4\end{pmatrix}$

so that $\boldsymbol{D}=\left|\frac{7(7)+2(1)+(-17)(-4)}{\sqrt{7^2+2^2+(-17)^2}}\right|=\left|\frac{119}{\sqrt{342}}\right|=\frac{119}{\sqrt{342}}$

(15) Find the shortest distance between the lines

$\frac{x-1}{2} = \frac{y+3}{5} = \frac{z-2}{3}$ and $\frac{x}{1} = \frac{y-4}{2} = \frac{z+1}{2}$, identifying the points on the lines at which this shortest distance occurs.

(15) Find the shortest distance between the lines

$\frac{x-1}{2}=\frac{y+3}{5}=\frac{z-2}{3}$ and $\frac{x}{1}=\frac{y-4}{2}=\frac{z+1}{2}$, identifying the points on the lines at which this shortest distance occurs.

Solution

Method 1

General points on the two lines are

$$\overrightarrow{OX}=\begin{pmatrix}1+2\lambda\\-3+5\lambda\\2+3\lambda\end{pmatrix} \text{ and } \overrightarrow{OY}=\begin{pmatrix}\mu\\4+2\mu\\-1+2\mu\end{pmatrix}$$

At the closest approach of the two lines, $\overrightarrow{XY}$ will be perpendicular to both lines, so that

$$\overrightarrow{XY}.\begin{pmatrix}2\\5\\3\end{pmatrix}=0 \text{ and } \overrightarrow{XY}.\begin{pmatrix}1\\2\\2\end{pmatrix}=0,$$

so that $\begin{pmatrix}\mu-(1+2\lambda)\\4+2\mu-(-3+5\lambda)\\-1+2\mu-(2+3\lambda)\end{pmatrix}.\begin{pmatrix}2\\5\\3\end{pmatrix}=0$ and

$$\begin{pmatrix}\mu-(1+2\lambda)\\4+2\mu-(-3+5\lambda)\\-1+2\mu-(2+3\lambda)\end{pmatrix}.\begin{pmatrix}1\\2\\2\end{pmatrix}=0,$$

giving $(2\mu-2-4\lambda)+(35+10\mu-25\lambda)+(-9+6\mu-9\lambda)=0$

or $18\mu-38\lambda=-24$; ie $9\mu-19\lambda=-12$ (1)

and $(\mu-1-2\lambda)+(14+4\mu-10\lambda)+(-6+4\mu-6\lambda)=0$

$9\mu-18\lambda=-7$ (2)

Then $(\boldsymbol{1})-(\boldsymbol{2}) \Rightarrow -\boldsymbol{\lambda}=-\boldsymbol{5}$, so that $\boldsymbol{\lambda}=\boldsymbol{5}$ and, from (2),

$\boldsymbol{\mu}=\frac{1}{9}(\boldsymbol{18}(\boldsymbol{5})-\boldsymbol{7})=\frac{83}{9}$

So $\overrightarrow{\boldsymbol{OX}}=\begin{pmatrix}11\\22\\17\end{pmatrix}$ and $\overrightarrow{\boldsymbol{OY}}=\frac{1}{9}\begin{pmatrix}83\\202\\157\end{pmatrix}$

and $\overrightarrow{\boldsymbol{XY}}=\frac{1}{9}\begin{pmatrix}83-99\\202-198\\157-153\end{pmatrix}=\frac{1}{9}\begin{pmatrix}-16\\4\\4\end{pmatrix}=\frac{4}{9}\begin{pmatrix}-4\\1\\1\end{pmatrix}$,

so that $\left|\overrightarrow{\boldsymbol{XY}}\right|=\frac{4}{9}\sqrt{16+1+1}=\frac{4\sqrt{18}}{9}=\frac{4\sqrt{2}}{3}$

Method 2

If $\underline{\hat{\boldsymbol{n}}}$ is a unit vector perpendicular to both lines, then we need $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ such that $\overrightarrow{\boldsymbol{OX}}+\boldsymbol{d}\underline{\hat{\boldsymbol{n}}}=\overrightarrow{\boldsymbol{OY}}$, and the shortest distance will then be $|\boldsymbol{d}|$.

A vector perpendicular to both lines is $\begin{pmatrix}2\\5\\3\end{pmatrix}\times\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{vmatrix}\underline{\boldsymbol{i}} & 2 & 1\\ \underline{\boldsymbol{j}} & 5 & 2\\ \underline{\boldsymbol{k}} & 3 & 2\end{vmatrix}$

$=\begin{pmatrix}4\\-1\\-1\end{pmatrix}$, so that $\underline{\hat{\boldsymbol{n}}}=\frac{1}{\sqrt{18}}\begin{pmatrix}4\\-1\\-1\end{pmatrix}$

Then $\overrightarrow{\boldsymbol{OX}}+\boldsymbol{d}\underline{\hat{\boldsymbol{n}}}=\overrightarrow{\boldsymbol{OY}}$ gives $\begin{pmatrix}1+2\lambda\\-3+5\lambda\\2+3\lambda\end{pmatrix}+\boldsymbol{D}\begin{pmatrix}4\\-1\\-1\end{pmatrix}=$
$\begin{pmatrix}\mu\\4+2\mu\\-1+2\mu\end{pmatrix}$,
where $\boldsymbol{D}=\frac{d}{\sqrt{18}}$,

so that $\boldsymbol{2\lambda + 4D - \mu = -1}$ (1)
$\boldsymbol{5\lambda - D - 2\mu = 7}$ (2)
$3\boldsymbol{\lambda - D - 2\mu = -3}$ (3)

Then $(\mathbf{2}) - (\mathbf{3}) \Rightarrow \boldsymbol{2\lambda = 10}$, so that $\boldsymbol{\lambda = 5}$

and (1) & (2) become $\boldsymbol{4D - \mu = -11}$ (4) and $\boldsymbol{-D - 2\mu = -18}$ (5)

Then $\mathbf{2}(\mathbf{4}) - (\mathbf{5}) \Rightarrow \boldsymbol{9D = -4}$, so that $|\boldsymbol{d}| = \sqrt{\mathbf{18}}|\boldsymbol{D}| = \frac{\mathbf{4}\sqrt{\mathbf{18}}}{\mathbf{9}} = \frac{\mathbf{4}\sqrt{\mathbf{2}}}{\mathbf{3}}$

and, from (1), $\boldsymbol{\mu = 10 - \frac{16}{9} + 1 = \frac{83}{9}}$

and $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ can then be found, as in (i).

Method 3

Suppose that the two closest points are

$$\overrightarrow{\boldsymbol{OX}} = \begin{pmatrix} \mathbf{1 + 2}\boldsymbol{\lambda} \\ \mathbf{-3 + 5}\boldsymbol{\lambda} \\ \mathbf{2 + 3}\boldsymbol{\lambda} \end{pmatrix} \text{ and } \overrightarrow{\boldsymbol{OY}} = \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{4 + 2}\boldsymbol{\mu} \\ \mathbf{-1 + 2}\boldsymbol{\mu} \end{pmatrix}$$

A vector perpendicular to both lines is $\begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{3} \end{pmatrix} \times \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{2} \end{pmatrix} = \begin{vmatrix} \underline{\boldsymbol{i}} & \mathbf{2} & \mathbf{1} \\ \underline{\boldsymbol{j}} & \mathbf{5} & \mathbf{2} \\ \underline{\boldsymbol{k}} & \mathbf{3} & \mathbf{2} \end{vmatrix}$

$= \begin{pmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix}$, and $\boldsymbol{Y}$ can be reached from $\boldsymbol{X}$ by travelling a certain distance in this direction.

Thus $\begin{pmatrix} \mathbf{1 + 2}\boldsymbol{\lambda} \\ \mathbf{-3 + 5}\boldsymbol{\lambda} \\ \mathbf{2 + 3}\boldsymbol{\lambda} \end{pmatrix} + \boldsymbol{k}\begin{pmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{4 + 2}\boldsymbol{\mu} \\ \mathbf{-1 + 2}\boldsymbol{\mu} \end{pmatrix}$, or $\begin{pmatrix} 2\lambda + 4k - \mu \\ 5\lambda - k - 2\mu \\ 3\lambda - k - 2\mu \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix}$

ie $\begin{pmatrix} 2 & 4 & -1 \\ 5 & -1 & -2 \\ 3 & -1 & -2 \end{pmatrix} \begin{pmatrix} \lambda \\ k \\ \mu \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \lambda \\ k \\ \mu \end{pmatrix} = \frac{1}{(0+45-27)} \begin{pmatrix} 0 & 4 & -2 \\ 9 & -1 & 14 \\ -9 & -1 & -22 \end{pmatrix}^T \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix}$

$= \frac{1}{18} \begin{pmatrix} 0 & 9 & -9 \\ 4 & -1 & -1 \\ -2 & 14 & -22 \end{pmatrix} \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix} = \frac{1}{18} \begin{pmatrix} 90 \\ -8 \\ 166 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 45 \\ -4 \\ 83 \end{pmatrix}$

Hence the two closest points are $\begin{pmatrix} 11 \\ 22 \\ 17 \end{pmatrix}$ and $\begin{pmatrix} \frac{83}{9} \\ \frac{202}{9} \\ \frac{157}{9} \end{pmatrix}$,

and the distance between them is $\left|-\frac{4}{9}\right| \sqrt{4^2 + (-1)^2 + (-1)^2}$

$= \frac{4\sqrt{18}}{9} = \frac{4\sqrt{2}}{3}$

(16)(i) Find the intersection of the line $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and the plane $3x + y + 4z = 77$

(ii) Find the shortest distance from the point $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ to the plane $3x + y + 4z = 77$

(16)(i) Find the intersection of the line $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and the plane $3x + y + 4z = 77$

(ii) Find the shortest distance from the point $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ to the plane

$3x + y + 4z = 77$

Solution

(i) For a point on the line, $x = 2 + 3\lambda, y = -1 + \lambda, z = 5 + 4\lambda$

Substituting into the eq'n of the plane:

$3(2 + 3\lambda) + (-1 + \lambda) + 4(5 + 4\lambda) = 77$

$\Rightarrow 26\lambda = 77 - 25 \Rightarrow \lambda = \frac{52}{26} = 2$

$\Rightarrow$ point of intersection has position vector $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 1 \\ 13 \end{pmatrix}$

(ii) From (i), nearest point on the plane is $\begin{pmatrix} 8 \\ 1 \\ 13 \end{pmatrix}$,

so that shortest distance is $\sqrt{(8-2)^2 + (1-[-1])^2 + (13-5)^2}$

$= \sqrt{36 + 4 + 64} = \sqrt{104} = 2\sqrt{26}$

Alternative method

From (i), shortest distance is $|\lambda||\underline{n}|$, where λ corresponds to the point of intersection of the line and plane, and $\underline{n}$ is the normal vector for the plane; ie $2\sqrt{3^2 + 1^2 + 4^2} = 2\sqrt{26}$

(17)(i) Given that the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$, what is the significance of $\frac{d}{|\underline{n}|}$ if $d > 0$?

(ii) Find the equation of the plane that is parallel to $\underline{r}.\underline{n} = d$ and contains the point $\underline{p}$.

(iii) Hence deduce the formula for the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$

(17)(i) Given that the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$, what is the significance of $\frac{d}{|\underline{n}|}$ if $d > 0$?

(ii) Find the equation of the plane that is parallel to $\underline{r}.\underline{n} = d$ and contains the point $\underline{p}$.

(iii) Hence deduce the formula for the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$

Solution

(i) $\frac{d}{|\underline{n}|}$ is the distance of the plane $\underline{r}.\underline{n} = d$ from the Origin,

when $d > 0$

(ii) $\underline{r}.\underline{n} = \underline{p}.\underline{n}$

(iii) The shortest distance from the plane $\underline{r}.\underline{n} = d$ to the Origin is $\frac{d}{|\underline{n}|}$.

The plane parallel to $\underline{r}.\underline{n} = d$, containing $\underline{p}$ has equation

$\underline{r}.\underline{n} = \underline{p}.\underline{n}$, and its shortest distance from the Origin is $\frac{\underline{p}.\underline{n}}{|\underline{n}|}$

Hence the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$

is $\left|\frac{\underline{\boldsymbol{p}}.\underline{\boldsymbol{n}}}{|\underline{\boldsymbol{n}}|} - \frac{\boldsymbol{d}}{|\underline{\boldsymbol{n}}|}\right|$

(18) Find the shortest distance between the point $(\mathbf{4}, -\mathbf{2}, \mathbf{3})$ and the line $\underline{\boldsymbol{r}} = \begin{pmatrix} \mathbf{7} \\ \mathbf{5} \\ -\mathbf{1} \end{pmatrix} + \boldsymbol{\lambda} \begin{pmatrix} \mathbf{3} \\ -\mathbf{6} \\ \mathbf{4} \end{pmatrix}$, leaving the answer in surd form.

(18) Find the shortest distance between the point $(\mathbf{4}, -\mathbf{2}, \mathbf{3})$ and the line $\underline{\boldsymbol{r}} = \begin{pmatrix} \mathbf{7} \\ \mathbf{5} \\ -\mathbf{1} \end{pmatrix} + \boldsymbol{\lambda} \begin{pmatrix} \mathbf{3} \\ -\mathbf{6} \\ \mathbf{4} \end{pmatrix}$, leaving the answer in surd form.

Solution

Method 1

Shortest distance $D = \frac{|(\underline{p}-\underline{a})\times\underline{d}|}{|\underline{d}|}$ where $\underline{p} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$, $\underline{a} = \begin{pmatrix} 7 \\ 5 \\ -1 \end{pmatrix}$ &

$\underline{d} = \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$

$$(\underline{p} - \underline{a}) \times \underline{d} = \begin{vmatrix} \underline{i} & -3 & 3 \\ \underline{j} & -7 & -6 \\ \underline{k} & 4 & 4 \end{vmatrix} = \begin{pmatrix} -4 \\ 24 \\ 39 \end{pmatrix}$$

So $D = \frac{\sqrt{(-4)^2+24^2+39^2}}{\sqrt{3^2+(-6)^2+4^2}} = \frac{\sqrt{2113}}{\sqrt{61}} = \sqrt{\frac{2113}{61}}$

Method 2

Let $\overrightarrow{OP} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$ and let M be the point on the line closest to P,

so that $\overrightarrow{OM} = \begin{pmatrix} 7+3\lambda \\ 5-6\lambda \\ -1+4\lambda \end{pmatrix}$, for some λ to be determined.

We require $\overrightarrow{PM}.\underline{d} = 0$,

so that $\begin{pmatrix} 7+3\lambda-4 \\ 5-6\lambda-(-2) \\ -1+4\lambda-3 \end{pmatrix}.\begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix} = 0$

$\Rightarrow 3(3+3\lambda) - 6(7-6\lambda) + 4(-4+4\lambda) = 0$

$\Rightarrow -49 + 61\lambda = 0 \Rightarrow \lambda = \frac{49}{61}$

The shortest distance $D = \left|\overrightarrow{PM}\right|$

where $\overrightarrow{PM} = \begin{pmatrix} 7+3\lambda-4 \\ 5-6\lambda-(-2) \\ -1+4\lambda-3 \end{pmatrix} = \begin{pmatrix} 3+3\lambda \\ 7-6\lambda \\ -4+4\lambda \end{pmatrix}$

So $D^2 = (3 + \frac{147}{61})^2 + (7 - \frac{294}{61})^2 + (-4 + \frac{196}{61})^2$

$= \frac{1}{61^2}(330^2 + 133^2 + (-48)^2)$

$= \frac{128893}{61^2} = \frac{2113}{61}$

and $D = \sqrt{\frac{2113}{61}}$

(19) Find the distance between the lines $\frac{\boldsymbol{x+1}}{\mathbf{1}} = \frac{\boldsymbol{y+2}}{\mathbf{2}}$; $\boldsymbol{z} = \mathbf{4}$ and $\frac{\boldsymbol{x+3}}{\mathbf{1}} = \frac{\boldsymbol{y-6}}{\mathbf{2}}$; $\boldsymbol{z} = \mathbf{7}$, leaving your answer in exact form.

(19) Find the distance between the lines $\frac{\boldsymbol{x+1}}{\mathbf{1}} = \frac{\boldsymbol{y+2}}{\mathbf{2}}; \boldsymbol{z} = \mathbf{4}$ and $\frac{\boldsymbol{x+3}}{\mathbf{1}} = \frac{\boldsymbol{y-6}}{\mathbf{2}}$; $\boldsymbol{z} = \mathbf{7}$, leaving your answer in exact form.

Solution

Method 1

The lines are parallel.

Choose a point on one of the lines; eg $\boldsymbol{P} = (\mathbf{-3}, \mathbf{6}, \mathbf{7})$ on the 2nd line.

To find the distance of this point from the 1st line:

A general point, Q on the 1st line is $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} \mathbf{-1} + \boldsymbol{\lambda} \\ \mathbf{-2} + \mathbf{2}\boldsymbol{\lambda} \\ \mathbf{4} \end{pmatrix}$

Then $\overrightarrow{\boldsymbol{PQ}} = \begin{pmatrix} \mathbf{-1} + \boldsymbol{\lambda} \\ \mathbf{-2} + \mathbf{2}\boldsymbol{\lambda} \\ \mathbf{4} \end{pmatrix} - \begin{pmatrix} \mathbf{-3} \\ \mathbf{6} \\ \mathbf{7} \end{pmatrix} = \begin{pmatrix} \mathbf{2} + \boldsymbol{\lambda} \\ \mathbf{-8} + \mathbf{2}\boldsymbol{\lambda} \\ \mathbf{-3} \end{pmatrix}$

We want $\overrightarrow{\boldsymbol{PQ}}$ to be perpendicular to the 1st line,

so that $\begin{pmatrix} \mathbf{2} + \boldsymbol{\lambda} \\ \mathbf{-8} + \mathbf{2}\boldsymbol{\lambda} \\ \mathbf{-3} \end{pmatrix} . \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{0} \end{pmatrix} = \mathbf{0}$

$\Rightarrow \mathbf{2} + \boldsymbol{\lambda} - \mathbf{16} + \mathbf{4}\boldsymbol{\lambda} = \mathbf{0} \quad \Rightarrow \mathbf{5}\boldsymbol{\lambda} = \mathbf{14}; \boldsymbol{\lambda} = \frac{\mathbf{14}}{\mathbf{5}}$

Then $\overrightarrow{\boldsymbol{PQ}} = \begin{pmatrix} \frac{\mathbf{24}}{\mathbf{5}} \\ -\frac{\mathbf{12}}{\mathbf{5}} \\ -\frac{\mathbf{15}}{\mathbf{5}} \end{pmatrix} = \frac{\mathbf{3}}{\mathbf{5}} \begin{pmatrix} \mathbf{8} \\ \mathbf{-4} \\ \mathbf{-5} \end{pmatrix}$ and the required distance is $\frac{\mathbf{3}}{\mathbf{5}}\sqrt{\mathbf{64} + \mathbf{16} + \mathbf{25}}$

$= \frac{3\sqrt{105}}{5}$

Method 2

Choose a point on each line; eg $\boldsymbol{R} = (-\mathbf{1}, -\mathbf{2}, \mathbf{4})$ on the 1st line, and
$\boldsymbol{P} = (-\mathbf{3}, \mathbf{6}, \mathbf{7})$ on the 2nd line.

Then $\overrightarrow{\boldsymbol{PR}} = \begin{pmatrix} \mathbf{2} \\ -\mathbf{8} \\ -\mathbf{3} \end{pmatrix}$ and the required distance is $\left| \frac{\begin{pmatrix} 2 \\ -8 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right|} \right|$

$$= \left| \frac{\begin{vmatrix} \underline{i} & 2 & 1 \\ \underline{j} & -8 & 2 \\ \underline{k} & -3 & 0 \end{vmatrix}}{\sqrt{5}} \right| = \frac{1}{\sqrt{5}} \left| \begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix} \right| = \frac{3}{\sqrt{5}} \left| \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right| = \frac{3}{\sqrt{5}} \sqrt{21} = \frac{3\sqrt{105}}{5}$$

(20)(i) Show the lines $\frac{\boldsymbol{x-1}}{\mathbf{2}}=\frac{\boldsymbol{y+3}}{\mathbf{5}}=\frac{\boldsymbol{z-2}}{\mathbf{3}}$ and $\frac{\boldsymbol{x}}{\mathbf{1}}=\frac{\boldsymbol{y-4}}{\mathbf{2}}=\frac{\boldsymbol{z+1}}{\mathbf{2}}$ are skew.

(ii) Find the shortest distance between the lines and identify the points on the lines at which this shortest distance occurs.

(20)(i) Show the lines $\frac{\boldsymbol{x-1}}{\mathbf{2}} = \frac{\boldsymbol{y+3}}{\mathbf{5}} = \frac{\boldsymbol{z-2}}{\mathbf{3}}$ and $\frac{\boldsymbol{x}}{\mathbf{1}} = \frac{\boldsymbol{y-4}}{\mathbf{2}} = \frac{\boldsymbol{z+1}}{\mathbf{2}}$ are skew.

(ii) Find the shortest distance between the lines and identify the points on the lines at which this shortest distance occurs.

Solution

(i) The lines can be rewritten in parametric form:

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \begin{pmatrix} \mathbf{1} + \mathbf{2}\boldsymbol{\lambda} \\ -\mathbf{3} + \mathbf{5}\boldsymbol{\lambda} \\ \mathbf{2} + \mathbf{3}\boldsymbol{\lambda} \end{pmatrix} \text{ and } \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{4} + \mathbf{2}\boldsymbol{\mu} \\ -\mathbf{1} + \mathbf{2}\boldsymbol{\mu} \end{pmatrix}$$

A point of intersection would then satisfy

$\mathbf{1} + \mathbf{2}\boldsymbol{\lambda} = \boldsymbol{\mu}$ (1)
$-\mathbf{3} + \mathbf{5}\boldsymbol{\lambda} = \mathbf{4} + \mathbf{2}\boldsymbol{\mu}$ (2)
$\mathbf{2} + \mathbf{3}\boldsymbol{\lambda} = -\mathbf{1} + \mathbf{2}\boldsymbol{\mu}$ (3)

Substituting from (1) into (2) & (3) gives:

$-\mathbf{3} + \mathbf{5}\boldsymbol{\lambda} = \mathbf{4} + \mathbf{2}(\mathbf{1} + \mathbf{2}\boldsymbol{\lambda})$ or $-\mathbf{9} = -\boldsymbol{\lambda}$, so that $\boldsymbol{\lambda} = \mathbf{9}$

and $\mathbf{2} + \mathbf{3}\boldsymbol{\lambda} = -\mathbf{1} + \mathbf{2}(\mathbf{1} + \mathbf{2}\boldsymbol{\lambda})$ or $\mathbf{1} = \boldsymbol{\lambda}$,

and so there is no point of intersection.

Also, the direction vectors of the lines are not parallel, and so the lines are skew.

(ii) [There are various methods for finding the shortest distance, but not all of them find the points on the lines where the shortest distance occurs. The first method given below is relatively straightforward, and doesn't involve the vector product.]

Method 1

From (i), general points on the two lines are

$$\overrightarrow{OX} = \begin{pmatrix} 1+2\lambda \\ -3+5\lambda \\ 2+3\lambda \end{pmatrix} \text{ and } \overrightarrow{OY} = \begin{pmatrix} \mu \\ 4+2\mu \\ -1+2\mu \end{pmatrix}$$

At the closest approach of the two lines, $\overrightarrow{XY}$ will be perpendicular to both lines, so that

$\overrightarrow{XY}.\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} = 0$ and $\overrightarrow{XY}.\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 0$, so that

$$\begin{pmatrix} \mu-(1+2\lambda) \\ 4+2\mu-(-3+5\lambda) \\ -1+2\mu-(2+3\lambda) \end{pmatrix}.\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} = 0 \text{ and}$$
$$\begin{pmatrix} \mu-(1+2\lambda) \\ 4+2\mu-(-3+5\lambda) \\ -1+2\mu-(2+3\lambda) \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 0,$$

giving $(2\mu-2-4\lambda)+(35+10\mu-25\lambda)+(-9+6\mu-9\lambda) = 0$
or $18\mu-38\lambda = -24$; ie $9\mu-19\lambda = -12$ (1)

and $(\mu-1-2\lambda)+(14+4\mu-10\lambda)+(-6+4\mu-6\lambda) = 0$

$9\mu-18\lambda = -7$ (2)

Then $(1)-(2) \Rightarrow -\lambda = -5$, so that $\lambda = 5$ and, from (2),

$$\mu = \frac{1}{9}(18(5)-7) = \frac{83}{9}$$

So $\overrightarrow{OX} = \begin{pmatrix} 11 \\ 22 \\ 17 \end{pmatrix}$ and $\overrightarrow{OY} = \frac{1}{9}\begin{pmatrix} 83 \\ 202 \\ 157 \end{pmatrix}$

and $\overrightarrow{\boldsymbol{XY}} = \frac{1}{9}\begin{pmatrix} \mathbf{83-99} \\ \mathbf{202-198} \\ \mathbf{157-153} \end{pmatrix} = \frac{1}{9}\begin{pmatrix} \mathbf{-16} \\ \mathbf{4} \\ \mathbf{4} \end{pmatrix} = \frac{4}{9}\begin{pmatrix} \mathbf{-4} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$,

so that $\left|\overrightarrow{\boldsymbol{XY}}\right| = \frac{4}{9}\sqrt{\mathbf{16+1+1}} = \frac{4\sqrt{18}}{9} = \frac{4\sqrt{2}}{3}$

Method 2 (using the vector product)

If $\underline{\hat{\boldsymbol{n}}}$ is a unit vector perpendicular to both lines, then we need $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ such that $\overrightarrow{\boldsymbol{OX}} + \boldsymbol{d}\underline{\hat{\boldsymbol{n}}} = \overrightarrow{\boldsymbol{OY}}$, and the shortest distance will then be $|\boldsymbol{d}|$.

A vector perpendicular to both lines is $\begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{3} \end{pmatrix} \times \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{2} \end{pmatrix} = \begin{vmatrix} \underline{\boldsymbol{i}} & \mathbf{2} & \mathbf{1} \\ \underline{\boldsymbol{j}} & \mathbf{5} & \mathbf{2} \\ \underline{\boldsymbol{k}} & \mathbf{3} & \mathbf{2} \end{vmatrix}$

$= \begin{pmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix}$, so that $\underline{\hat{\boldsymbol{n}}} = \frac{1}{\sqrt{18}}\begin{pmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix}$

Then $\overrightarrow{\boldsymbol{OX}} + \boldsymbol{d}\underline{\hat{\boldsymbol{n}}} = \overrightarrow{\boldsymbol{OY}}$ gives $\begin{pmatrix} \mathbf{1+2}\boldsymbol{\lambda} \\ \mathbf{-3+5}\boldsymbol{\lambda} \\ \mathbf{2+3}\boldsymbol{\lambda} \end{pmatrix} + \boldsymbol{D}\begin{pmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix} =$

$\begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{4+2}\boldsymbol{\mu} \\ \mathbf{-1+2}\boldsymbol{\mu} \end{pmatrix}$,

where $\boldsymbol{D} = \frac{\boldsymbol{d}}{\sqrt{18}}$,

so that $\mathbf{2}\boldsymbol{\lambda} + \mathbf{4}\boldsymbol{D} - \boldsymbol{\mu} = \mathbf{-1}$ (1)
$\mathbf{5}\boldsymbol{\lambda} - \boldsymbol{D} - \mathbf{2}\boldsymbol{\mu} = \mathbf{7}$ (2)
$3\boldsymbol{\lambda} - \boldsymbol{D} - \mathbf{2}\boldsymbol{\mu} = \mathbf{-3}$ (3)

Then $(\mathbf{2}) - (\mathbf{3}) \Rightarrow \mathbf{2}\boldsymbol{\lambda} = \mathbf{10}$, so that $\boldsymbol{\lambda} = \mathbf{5}$

and (1) & (2) become $\boldsymbol{4D-\mu=-11}$ (4) and $\boldsymbol{-D-2\mu=-18}$ (5)

Then $\boldsymbol{2(4)-(5)\Rightarrow 9D=-4}$, so that $|\boldsymbol{d}|=\sqrt{\mathbf{18}}|\boldsymbol{D}|=\frac{\mathbf{4\sqrt{18}}}{\mathbf{9}}=\frac{\mathbf{4\sqrt{2}}}{\mathbf{3}}$

and, from (1), $\boldsymbol{\mu}=\mathbf{10}-\frac{\mathbf{16}}{\mathbf{9}}+\mathbf{1}=\frac{\mathbf{83}}{\mathbf{9}}$

and $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ can then be found, as in (i).

(21) Show that the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$

(21) Show that the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$

Solution

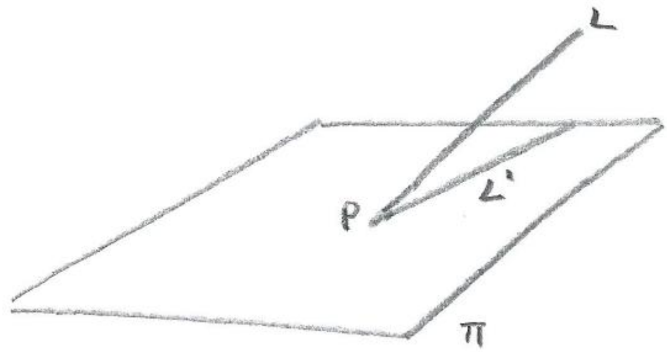
$(\underline{p} + \lambda\underline{n}).\underline{n} = d \Rightarrow \underline{p}.\underline{n} + \lambda|\underline{n}|^2 = d$

$\Rightarrow \lambda = \frac{d-\underline{p}.\underline{n}}{|\underline{n}|^2}$

So shortest distance $= |\lambda||\underline{n}| = \frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$

(22) Given the plane Π: $3x + 2y - z = 6$ and the line

$L: \underline{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, let L' be the projection of L onto Π



(i) Find the point of intersection (P) of Π & L

(ii) Find the angle between Π & L

(iii) Find a vector that is parallel to Π and perpendicular to L

(iv) Find a vector equation for L'

(v) Find the angle between L and L'

Solution

(i) $3(1+2\lambda)+2(-\lambda)-(3+\lambda)=6$

$\Rightarrow 3\lambda=6 \Rightarrow \lambda=2$

So P is $\begin{pmatrix}1+4\\-2\\3+2\end{pmatrix}=\begin{pmatrix}5\\-2\\5\end{pmatrix}$

(ii) The angle between Π & L is the angle between L and its projection onto the plane (ie the angle between L and L'), but is most easily determined by first finding the angle between L and the normal to the plane, and subtracting this from $\frac{\pi}{2}$

If θ is the angle between L and the normal to the plane, then

$$cos\theta=\frac{\begin{pmatrix}2\\-1\\1\end{pmatrix}.\begin{pmatrix}3\\2\\-1\end{pmatrix}}{\sqrt{4+1+1}\sqrt{9+4+1}}=\frac{6-2-1}{\sqrt{6}\sqrt{14}}=\frac{3}{2\sqrt{21}}=\frac{3\sqrt{21}}{2(21)}=\frac{\sqrt{21}}{14}$$

The required angle is then arcsin $\left(\frac{\sqrt{21}}{14}\right)=19.107°=19.1°$ (1dp)

(iii) [Note that "parallel to the plane" means parallel to a vector in the plane, and therefore perpendicular to the normal to the plane. The required vector is also perpendicular to the plane containing L and L'.]

As the required vector is perpendicular to both the normal to the plane and L, we can use the vector product:

$$\begin{pmatrix}3\\2\\-1\end{pmatrix}\times\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{vmatrix}\underline{i} & 3 & 2\\ \underline{j} & 2 & -1\\ \underline{k} & -1 & 1\end{vmatrix}=\underline{i}-5\underline{j}-7\underline{k}$$

[A useful check is that the scalar product with the original vectors is zero. Thus $3(1) + 2(-5) + (-1)(-7) = 0$]

(iv) L' passes through P and its direction is perpendicular to both

$\underline{i} - 5\underline{j} - 7\underline{k}$ (from (iii)) and the normal to the plane.

So its direction vector is

$$\begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{vmatrix} \underline{i} & 1 & 3 \\ \underline{j} & -5 & 2 \\ \underline{k} & -7 & -1 \end{vmatrix} = 19\underline{i} - 20\underline{j} + 17\underline{k}$$

So a vector equation of L' is: $\underline{r} = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 19 \\ -20 \\ 17 \end{pmatrix}$, from (i).

(v) If the required angle is ϕ, then

$$cos\phi = \frac{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} . \begin{pmatrix} 19 \\ -20 \\ 17 \end{pmatrix}}{\sqrt{4+1+1}\sqrt{361+400+289}} = \frac{38+20+17}{\sqrt{6}\sqrt{1050}}$$

$$= \frac{75}{2\sqrt{3}\sqrt{525}} = \frac{75}{2\sqrt{3}(5)\sqrt{21}} = \frac{15}{2(3)\sqrt{7}} = \frac{5\sqrt{7}}{14}$$

and hence $\phi = \arccos\left(\frac{5\sqrt{7}}{14}\right) = 19.107° = 19.1°$ (1dp) (as in (ii))