Vectors - Important Ideas (18 pages; 21/1/21)

Contents

(1) Vector equation of a line

(i) Note the distinction between:

(a) the vector equation of the line passing through A and B (sometimes abbreviated to "... the line AB"; though strictly speaking it should read "... the line segment AB extended"), which is the position vector of a general point on the line, and

(b) the vector $\overrightarrow{AB}$, which is a vector in the direction of the line

(ii) If A and B are points on the line, and $\underline{d}$ is the direction of the line, then the following forms of the vector equation are possible:

$\underline{r} = \underline{a} + \lambda\underline{d}$ (where $\underline{a} = \overrightarrow{OA}$)

$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$

$\underline{r} = (1 - \lambda)\underline{a} + \lambda\underline{b}$

[this can be considered to be a weighted average of $\underline{a}$ and $\underline{b}$]

(iii) When asked for the vector equation of a line, it is essential to include the "$\underline{r} =$". Note that $\underline{r}$ can be replaced by $\begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, as appropriate, and that the vector equation can be written as 2 or 3 scalar equations.

(2) Cartesian form of a line in 3D

(a) The line $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ can be written as

$(\lambda =)\ \frac{x-2}{3} = \frac{y-4}{5} = \frac{z-6}{2}$

[More generally, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} a_1 + \lambda d_1 \\ a_2 + \lambda d_2 \\ a_3 + \lambda d_3 \end{pmatrix}$

becomes $\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$]

(b) The line $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$ would be written as

$\frac{x-2}{3} = \frac{y-4}{5}$, $z = 6$ (as $\frac{z-6}{0}$ is undefined)

It represents a line in the plane $z = 6$.

(c) The line $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ could be written as

$\lambda = \frac{x-2}{3}, y = 4\ , z = 6$

It represents the line parallel to the x-axis passing through the point (0,4,6).

As x can take any value, the form $x = k, y = 4\ , z = 6$ is preferable.

(3) Scalar product

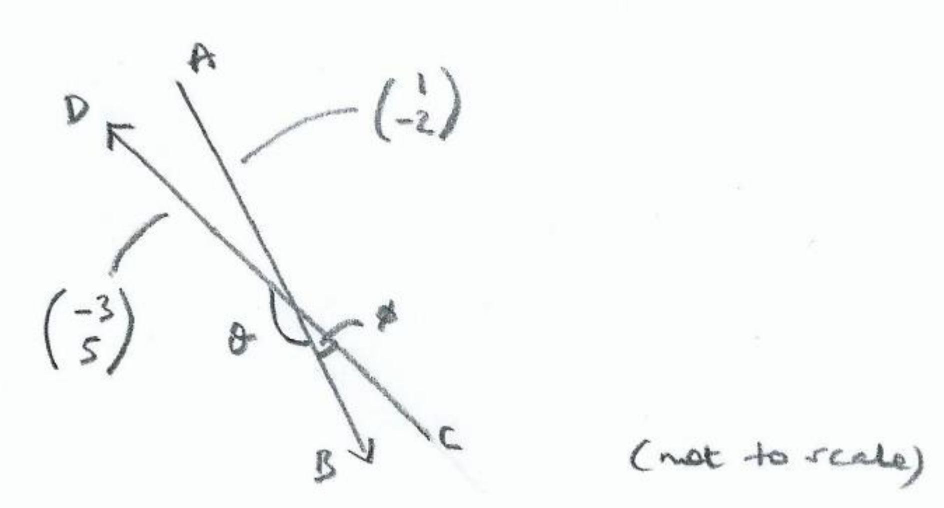
(i) $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \left(a_1\underline{i} + a_2\underline{j}\right) . \left(b_1\underline{i} + b_2\underline{j}\right)$

$= a_1b_1\underline{i}.\underline{i} + a_1b_2\underline{i}.\underline{j} + a_2b_1\underline{j}.\underline{i} + a_2b_2\underline{j}.\underline{j}$

$= a_1b_1 + 0 + 0 + a_2b_2$

$= a_1b_1 + a_2b_2$

(ii) Consider two line segments, $\overrightarrow{AB} = \binom{1}{-2}$ and $\overrightarrow{CD} = \binom{-3}{5}$



[the locations of the lines are not important; only their directions]

Note that the gradients of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are $\frac{-2}{1} = -2$ and $\frac{5}{-3} = -\frac{5}{3}$.

We can find the angle θ between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ as follows:

$\overrightarrow{AB}.\overrightarrow{CD} = |\overrightarrow{AB}||\overrightarrow{CD}|cos\theta$,

giving $\binom{1}{-2}.\binom{-3}{5} = \sqrt{1^2 + (-2)^2}\sqrt{(-3)^2 + 5^2}\, cos\theta$

so that $cos\theta = \frac{-3-10}{\sqrt{5}\sqrt{34}} = \frac{-13}{\sqrt{170}}$

and hence $\theta = 175.601° = 175.6°$ (1dp)

Now consider the angle ϕ between $\overrightarrow{AB}$ and $\overrightarrow{DC} = -\binom{-3}{5} = \binom{3}{-5}$

(note that the gradient of $\overrightarrow{DC}$ is still $-\frac{5}{3}$).

Then $\overrightarrow{AB}.\overrightarrow{DC} = |\overrightarrow{AB}||\overrightarrow{DC}|cos\phi = |\overrightarrow{AB}||\overrightarrow{CD}|cos\phi$, so that $\begin{pmatrix} 1 \\ -2 \end{pmatrix} . \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \sqrt{5}\sqrt{34}\, cos\phi$

and $cos\phi = \frac{3+10}{\sqrt{5}\sqrt{34}} = \frac{13}{\sqrt{170}}$

and hence $\phi = 180 - 175.6° = 4.4°$ (1dp)

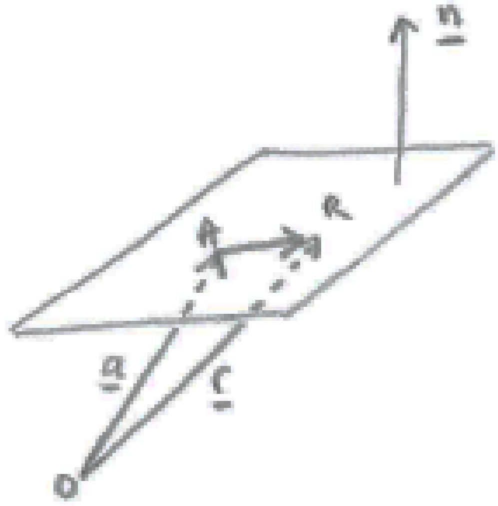
This is consistent with the diagram above.

Note that, if asked to find the angle between the two lines, without any directions being specified (ie whether $\overrightarrow{AB}$ or $\overrightarrow{BA}$), it is customary to give the acute angle; ie 4.4° in this case.

(iii) $\underline{a}.\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = {a_1}^2 + {a_2}^2 + {a_3}^2 = |\underline{a}|^2$

(4) Equation of a plane

(a) Vector and Cartesian forms



Referring to the diagram, where A is a given point in the plane, and R is a general point in the plane (with position vectors $\underline{a}$ and $\underline{r}$, respectively), if $\underline{n}$ is a normal to the plane, then $\underline{r} - \underline{a}$ will be perpendicular to $\underline{n}$,

so that $(\underline{r} - \underline{a}).\underline{n} = 0$

$\Rightarrow \underline{r}.\underline{n} = \underline{a}.\underline{n} = d$ (say)

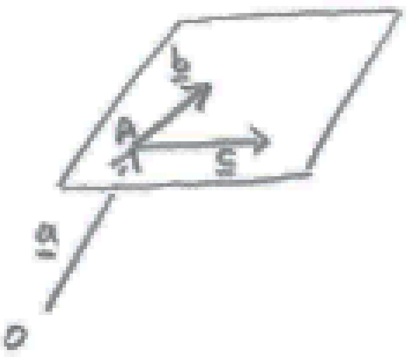
and so $\underline{r}.\underline{n} = d$, which is the 'vector' form of the eq'n of the plane

$\Rightarrow n_x x + n_y y + n_z z = d$, which is the 'Cartesian' form

[Alternatively, if $\underline{a}$ lies in the plane $\underline{r}.\underline{n} = d$,

then $\underline{a}.\underline{n} = d$, so that $\underline{r}.\underline{n} = \underline{a}.\underline{n}$]

(b) Parametric form



Referring to the diagrams, if $\underline{b}$ & $\underline{c}$ are direction vectors in the plane, then $\underline{r} = \underline{a} + \lambda\underline{b} + \mu\underline{c}$

Notes

(i) If B & C are two further points in the plane (such that A, B & C are not on a straight line), then $\underline{b}$ & $\underline{c}$ can be obtained from

$\overrightarrow{OB} - \overrightarrow{OA}$ and $\overrightarrow{OC} - \overrightarrow{OA}$, respectively.

(ii) The vector form can be obtained by taking $\underline{n} = \underline{b} \times \underline{c}$

(Alternatively, eliminate λ and μ from the 3 scalar eq'ns making up $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, and hence form the Cartesian equation.)

(iii) To convert from the Cartesian to the parametric form, let

$x = s$ and $y = t$, to find z in terms of s and t, and giving

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ? \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ ? \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ ? \end{pmatrix}$$

(c) 'Unit normal' form [my terminology]

If the vector form is $\underline{r}.\underline{n} = d$, and if $\underline{\hat{n}} = \frac{\underline{n}}{|\underline{n}|}$ (so that $\underline{\hat{n}}$ has unit magnitude), then $\underline{r}.\underline{\hat{n}} = \frac{d}{|\underline{n}|}$, and $\frac{d}{|\underline{n}|}$ can be shown to be the shortest (perpendicular) distance of the plane from the Origin.

Example

Let P be the plane $2x - 4y + z = 3$

To find the distance of P from the Origin, consider the line L that is perpendicular to P and passes through the Origin. Its equation is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$

The intersection of L and P is obtained by substituting for x, y & z in the eq'n of the plane, to give

$$2(2\lambda) - 4(-4\lambda) + (\lambda) = 3 \Rightarrow \lambda = \frac{3}{|\underline{n}|^2},$$

where $|\underline{n}| = \left| \begin{matrix} 2 \\ -4 \\ 1 \end{matrix} \right| = \sqrt{2^2 + (-4)^2 + 1^2}$

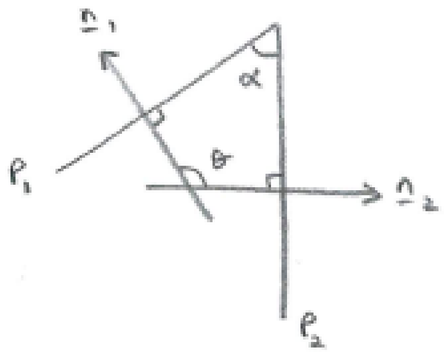
Then the distance of P from the Origin is the distance travelled along $L = \lambda \left| \begin{matrix} 2 \\ -4 \\ 1 \end{matrix} \right| = \frac{3}{|\underline{n}|^2} |\underline{n}| = \frac{3}{|\underline{n}|}$

(5) Angle between two planes

The angle between two planes is the acute angle between the

the lines having the direction vectors of the normals of the planes.

This can be seen as follows:



Referring to the example in the diagram, θ is the angle between the two normals $\underline{n}_1$ and $\underline{n}_2$, and can be found from their scalar product. If θ is obtuse (as in this case), then the angle between the planes is $\alpha = 180 - \theta$. This is the acute angle between the lines having the direction vectors of the normals.

[Note that, whilst the angle between the two normals is obtuse (as each normal is pointing in a particular direction), if the normals are replaced with lines (with no arrows), then there are two possibilities for 'the angle between the lines': either the acute angle, or the obtuse angle.]

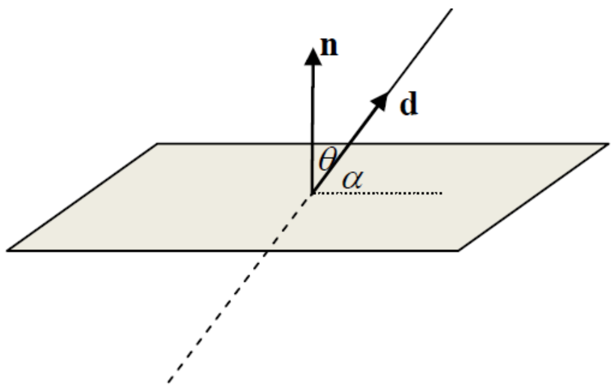
If either of the normals in the diagram is reversed in direction, then the angle between the normals is acute, and equals $180 - \theta$.

Once again, α equals the acute angle between the lines having the direction vectors of the normals.

(If both of the normals are reversed in direction, then the angle between the normals will equal θ again, as in the first case.)

Thus, in all cases, the angle between the planes is equal to the acute angle between the lines having the direction vectors of the normals.

(6) Angle between a line and a plane



To determine the angle between a line (with direction $\underline{d}$) and a plane (with normal $\underline{n}$):

(i) Find the acute angle θ between the lines with the direction vectors of $\underline{d}$ & $\underline{n}$

(ii) Subtract θ from 90°, to give the required angle α

[Note: Referring to the diagram, if we reverse either $\underline{d}$ or $\underline{n}$, then the angle between $\underline{d}$ and $\underline{n}$ becomes obtuse, and the required acute angle is obtained by subtracting this obtuse angle from 180°.]

(7) Vector perpendicular to a given (2D) vector

$\begin{pmatrix} -b \\ a \end{pmatrix}$ is perpendicular to $\begin{pmatrix} a \\ b \end{pmatrix}$

(8) Intersection of two lines

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$:

eliminate λ and μ from $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$

Note: If no solution exists (ie if the equations are not consistent), then the lines are skew.

(9) Intersection of a line and a plane

Point of intersection of the line $\underline{r} = \underline{a} + \lambda \underline{d}$ and the plane $\underline{r}.\underline{n} = b.\underline{n} \Rightarrow (\underline{a} + \lambda \underline{d}).\underline{n} = b.\underline{n}$, giving a value for λ, and hence the required point on the line.

(10) Line of intersection of two planes

Method 1

Substitute $x = \lambda$ into the cartesian equations of the two planes, and find y and z in terms of λ, to give

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ ? \\ ? \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ ? \\ ? \end{pmatrix}$$

Method 2

Find two points, $\underline{a}$ and $\underline{b}$, that lie on both of the planes (and hence on the line); eg by setting $x = 0$ (for one point) and $y = 0$ (for another).

The equation of the intersecting line is then $\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$

Method 3

Find a point that lies on both of the planes; then for the direction of the line, take the vector product of the normals of the two planes (as the line will be perpendicular to both of these).

(11) Shortest distance from a point to a plane

To find the shortest distance from the point $\underline{p}$ to the plane

$\underline{r}.\underline{n} = d$:

Method 1

Obtain the unit normal vector $\underline{\hat{n}} = \frac{\underline{n}}{|\underline{n}|}$

and rewrite $\underline{r}.\underline{n} = d$ as $\underline{r}.\underline{\hat{n}} = d'$, where $d' = \frac{d}{|\underline{n}|}$

Then consider the line $\underline{r} = \underline{p} + \lambda\underline{\hat{n}}$, and the point where this meets the plane; ie where $(\underline{p} + \lambda\underline{\hat{n}}).\underline{\hat{n}} = d'$

The value of λ obtained from this eq'n gives the required distance: $|\lambda|$.

Method 2

Create the equation of the plane passing through $\underline{p}$, parallel to the plane $\underline{r}.\underline{\hat{n}} = d'$, to give $\underline{r}.\underline{\hat{n}} = e'$

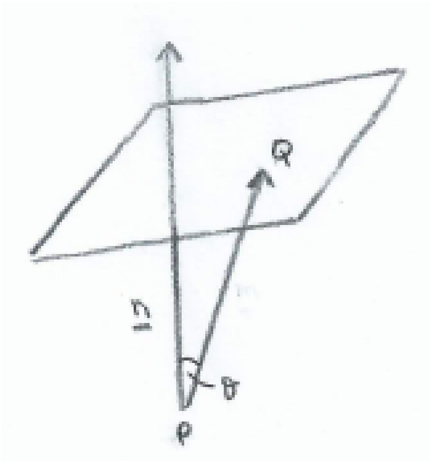
Then the required distance is $|d' - e'|$

Note: This is how the standard formula $\frac{|n_1p_1+n_2p_2+n_3p_3-d|}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$ is derived:

$d' = \frac{d}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$ and $e' = \underline{p}.\underline{\hat{n}} = \frac{n_1p_1+n_2p_2+n_3p_3}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$

Method 3

Find any point Q in the plane (eg by setting $x = y = 0$ in the cartesian form).



The required distance will then be the projection of $\overrightarrow{PQ}$ onto $\underline{n}$ (the normal to the plane); namely $\frac{|\overrightarrow{PQ}.\underline{n}|}{|\underline{n}|}$

(12) Distance between two parallel planes

As for the shortest distance from a point to a plane, if the two planes are written in the form $\underline{r}.\underline{\hat{n}} = d'$ and $\underline{r}.\underline{\hat{n}} = e'$

(where $\underline{\hat{n}} = \frac{\underline{n}}{|\underline{n}|}$ is the unit normal vector, and $d' = \frac{d}{|\underline{n}|}$ (and similarly for e')),

then the required distance is $|d' - e'|$

(13) Vector product

$$\underline{a} \times \underline{b} = \left(a_1\underline{i} + a_2\underline{j} + a_3\underline{k}\right) \times \left(b_1\underline{i} + b_2\underline{j} + b_3\underline{k}\right)$$

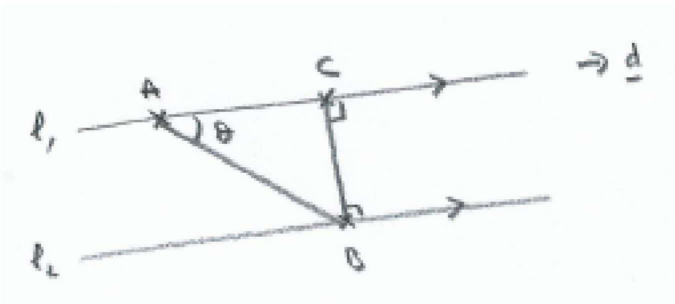
$$= (a_2b_3 - a_3b_2)\underline{i} + (a_3b_1 - a_1b_3)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\underline{k}$$

$$= \begin{vmatrix} \underline{i} & a_1 & b_1 \\ \underline{j} & a_2 & b_2 \\ \underline{k} & a_3 & b_3 \end{vmatrix}$$ (or the transpose of this)

(14) Shortest distance from a point to a line / distance between parallel lines

If A and B are given points on the two lines, and $\underline{d}$ is the common direction vector:



Method 1

Let C be the point on l_1 with parameter k, so that $\underline{c} = \underline{a} + k\underline{d}$ (*)

Then we require $\underline{d}.\left(\underline{c} - \underline{b}\right) = 0$

Solving this equation for k and substituting for k in (*) gives $\underline{c}$, and the distance between the two lines is then $|\underline{c} - \underline{b}|$.

Method 2

Having obtained the general point, C on l_1 in Method 1, we can minimise the distance BC by finding the stationary point of BC^2 (ie where $\frac{d}{dk}\ (BC^2) = 0$)

Method 3

As $BC = ABsin\theta,\ BC = \frac{|\overrightarrow{AB} \times \underline{d}|}{|\underline{d}|}$

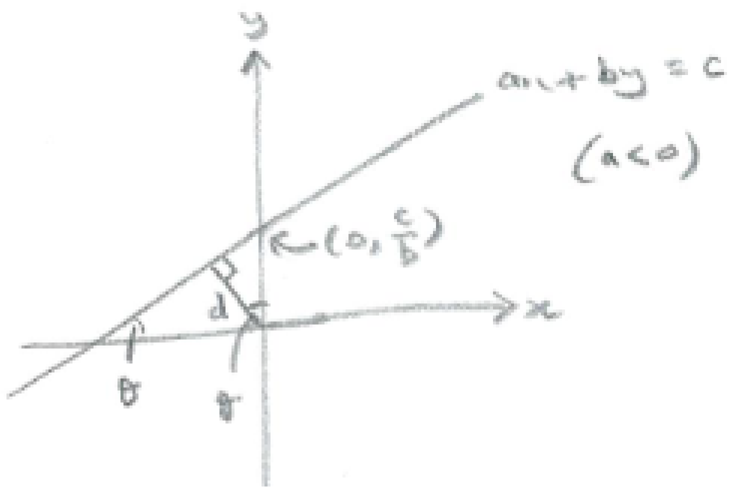
Method 4 (2D lines)

The equivalent of the formula $\frac{|n_1b_1+n_2b_2+n_3b_3-d|}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$ for the shortest distance from a point to a plane (see above) gives

$\frac{|ax_1+by_1-c|}{\sqrt{a^2+b^2}}$ as the shortest distance from the point (x_1, y_1) to the line $ax + by = c$

Demonstration

We first establish the distance of a 2D line from the Origin.



Referring to the diagram, $ax + by = c \Rightarrow y = \frac{c}{b} - \frac{a}{b}x$, so that $tan\theta = -\frac{a}{b}$

and $d = \frac{c}{b}\ cos\theta = \frac{c}{b}.\frac{1}{\sqrt{tan^2\theta+1}} = \frac{c}{b}\ .\frac{1}{\sqrt{\frac{a^2}{b^2}+1}} = \frac{c}{\sqrt{a^2+b^2}}$

So if the line $ax + by = c$ is written in the form

$\frac{1}{\sqrt{a^2+b^2}}(ax + by) = \frac{c}{\sqrt{a^2+b^2}}$, then the right-hand side is the

distance of the line from the Origin.

Now consider the line through the point (x_1, y_1) that is parallel to

$ax + by = c$:

Suppose that it has eq'n $ax + by = c'$ (as this line has the required gradient).

Then $ax_1 + by_1 = c'$, and so the required eq'n is

$ax + by = ax_1 + by_1$

and its distance from the Origin is $\frac{ax_1+by_1}{\sqrt{a^2+b^2}}$

Hence the shortest distance from the point (x_1, y_1) to the line
$ax + by = c$ (which equals the distance between the two lines)

equals $\left|\frac{ax_1+by_1}{\sqrt{a^2+b^2}} - \frac{c}{\sqrt{a^2+b^2}}\right| = \frac{|ax_1+by_1-c|}{\sqrt{a^2+b^2}}$

(15) Vector product form of a line

$(\underline{r} - \underline{a}) \times \underline{d} = \underline{0}$ or $\underline{r} \times \underline{d} = \underline{a} \times \underline{d}$

(16) Vector perpendicular to two vectors

To find a vector perpendicular to the (3D) vectors $\underline{a}$ and $\underline{b}$:

Method 1

$\underline{a} \times \underline{b}$

Method 2

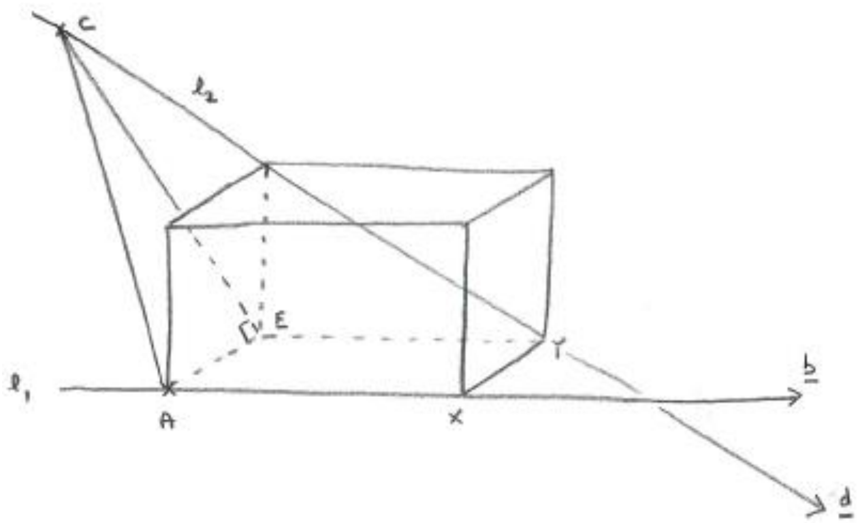
Let $\underline{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ be the required vector.

Then eliminate two of d_1, d_2 & d_3 from $\underline{d}.\underline{a} = 0$ and $\underline{d}.\underline{b} = 0$ (*)

to give a direction vector in terms of parameter d_1, d_2 or d_3.

eg $\begin{pmatrix} d_1 \\ 2d_1 \\ 3d_1 \end{pmatrix}$, which is equivalent to the direction vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(17) Shortest distance between two skew lines



(l_1: $\underline{r} = \underline{a} + \lambda\underline{b}$ & l_2: $\underline{r} = \underline{c} + \mu\underline{d}$)

Method 1

$|(\underline{c} - \underline{a}).\frac{(\underline{b}\times\underline{d})}{|\underline{b}\times\underline{d}|}|$

Justification

If a cuboid is drawn so that l_1 lies along one edge, as shown in the diagram, then by a choice of a suitable length, width and depth for the cuboid, l_2 will lie along the diagonal of the back face, as shown.

XY is then the required distance (with $\overrightarrow{XY}$ being perpendicular to both lines), and this is equal to AE.

Triangle AEC is right-angled,

and $AE = AC cos C\hat{A}E = \left|\frac{\overrightarrow{AC}.\overrightarrow{AE}}{|\overrightarrow{AE}|}\right|$

Now, $\underline{b} \times \underline{d}$ is perpendicular to both $\underline{b}$ & $\underline{d}$, and therefore has the same direction as $\overrightarrow{XY}$ and $\overrightarrow{AE}$, so that $\frac{\overrightarrow{AC}.\overrightarrow{AE}}{|\overrightarrow{AE}|} = \frac{\overrightarrow{AC}.(\underline{b}\times\underline{d})}{|(\underline{b}\times\underline{d})|}$

and $AE = |(\underline{c} - \underline{a}).\frac{(\underline{b}\times\underline{d})}{|\underline{b}\times\underline{d}|}|$

Note: Two lines in 3D will intersect if $(\underline{c} - \underline{a}).(\underline{b} \times \underline{d}) = 0$

Method 2

Suppose that X and Y have position vectors $\underline{r} = \underline{a} + \lambda_X \underline{b}$ and

$\underline{r} = \underline{c} + \mu_Y \underline{d}$, so that $\overrightarrow{XY} = (\underline{c} + \mu_Y \underline{d}) - (\underline{a} + \lambda_X \underline{b})$

As $\overrightarrow{XY}$ is perpendicular to both lines,

$\overrightarrow{XY}.\underline{b} = \overrightarrow{XY}.\underline{d} = 0$,

and solving these eq'ns enables λ_X & μ_Y , and hence $|\overrightarrow{XY}|$, to be determined.

Method 3

Again, suppose that X and Y have position vectors

$\underline{r} = \underline{a} + \lambda_X \underline{b}$ and $\underline{r} = \underline{c} + \mu_Y \underline{d}$

Then $\overrightarrow{OY} = \overrightarrow{OX} + \overrightarrow{XY}$, so that

$\underline{c} + \mu_Y \underline{d} = \underline{a} + \lambda_X \underline{b} + k\underline{n}$ (*)

where $\underline{n} = \underline{b} \times \underline{d}$ (as $\underline{b} \times \underline{d}$ is perpendicular to both $\underline{b}$ & $\underline{d}$, and therefore has the same direction as $\overrightarrow{XY}$), and then $XY = k|\underline{n}|$

Then (*) gives 3 simultaneous equations in λ_X, μ_Y & k:

$\begin{pmatrix} c_1 + \mu_Y d_1 \\ c_2 + \mu_Y d_2 \\ c_3 + \mu_Y d_3 \end{pmatrix} = \begin{pmatrix} a_1 + \lambda_X b_1 + kn_1 \\ a_2 + \lambda_X b_2 + kn_2 \\ a_3 + \lambda_X b_3 + kn_3 \end{pmatrix}$, from which k can be found,

and then $XY = k\left|\underline{b} \times \underline{d}\right|$