Vectors Exercises - Part 1 (Sol'ns) (25 pages; 19/5/20)

(1*) Vector equation of line

Given that the line $\underline{r} = \binom{2}{3} + \lambda\binom{1}{-2}$ can also be written as

$\binom{0}{7} + \mu\binom{-1}{2}$, find μ in terms of λ

Solution

$\binom{0}{7} + \mu\binom{-1}{2} = \binom{2}{3} + \binom{-2}{4} + \mu\binom{-1}{2}$

$= \binom{2}{3} + 2\binom{-1}{2} + \mu\binom{-1}{2}$

$= \binom{2}{3} + (2+\mu)\binom{-1}{2}$

Thus $2 + \mu = -\lambda$, and so $\mu = -\lambda - 2$

(2*) Vector equation of line

Find a vector equation of the line that passes through the point (1,2) and is perpendicular to the line $\underline{r} = \binom{3}{4} + \lambda\binom{4}{-1}$

Solution

Method 1

The gradient of the given line is $\frac{-1}{4}$, so that the gradient of the perpendicular line is 4.

Then a vector equation of the required line is

$\underline{r} = \binom{1}{2} + \lambda\binom{1}{4}$

Method 2 (much longer, but good practice!)

Let P be the intersection of the given line (L, say) and the perpendicular line through Q(1,2). Then P can be represented as $\begin{pmatrix} 3+4\lambda \\ 4-\lambda \end{pmatrix}$, for some λ to be determined.

Then, as L is perpendicular to QP, $\begin{pmatrix} 4 \\ -1 \end{pmatrix} . \begin{pmatrix} 3+4\lambda-1 \\ 4-\lambda-2 \end{pmatrix} = 0$

[noting that $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ is the direction vector of L; not to be confused with $\begin{pmatrix} 3+4\lambda \\ 4-\lambda \end{pmatrix}$, which the position vector of a point on L]

so that $4(2+4\lambda) - (2-\lambda) = 0$,

and hence $17\lambda + 6 = 0$, and $\lambda = -\frac{6}{17}$

Thus P is $\begin{pmatrix} 3+4\left(-\frac{6}{17}\right) \\ 4-\left(-\frac{6}{17}\right) \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 27 \\ 74 \end{pmatrix}$

And a vector equation of the line through P and Q is

$\underline{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda[\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{17}\begin{pmatrix} 27 \\ 74 \end{pmatrix}]$

or $\underline{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{\lambda}{17}\begin{pmatrix} 17-27 \\ 34-74 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{\lambda}{17}\begin{pmatrix} -10 \\ -40 \end{pmatrix}$

or $\underline{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(3**) Scalar product

Show that if $|\underline{a} - \underline{b}| = |\underline{a} + \underline{b}|$, then $\underline{a}$ & $\underline{b}$ are perpendicular.

Solution

$|\underline{a}-\underline{b}| = |\underline{a}+\underline{b}| \Rightarrow |\underline{a}-\underline{b}|^2 = |\underline{a}+\underline{b}|^2$

$\Rightarrow (\underline{a}-\underline{b}).(\underline{a}-\underline{b}) = (\underline{a}+\underline{b}).(\underline{a}+\underline{b})$

$[\ \underline{x}.\underline{x} = |\underline{x}|\,|\underline{x}|cos0° = |\underline{x}|^2\ ]$

$\Rightarrow \underline{a}.\underline{a} - \underline{a}.\underline{b} - \underline{b}.\underline{a} + \underline{b}.\underline{b} = \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{b}.\underline{a} + \underline{b}.\underline{b}$

$\Rightarrow -2\underline{a}.\underline{b} = 2\underline{a}.\underline{b}$ [since $\underline{a}.\underline{b} = \underline{b}.\underline{a}$]

$\Rightarrow \underline{a}.\underline{b} = 0$

and hence $\underline{a}$ & $\underline{b}$ are perpendicular

[Geometrically, $|\underline{a}-\underline{b}|$&$|\underline{a}+\underline{b}|$ are the 'short' and 'long' diagonals of the parallelogram formed from the adjacent sides $\underline{a}$ & $\underline{b}$. When these diagonals are equal, the parallelogram is a rectangle.]

(4) Planes**

Find the cartesian form of the plane

$$\underline{r} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Solution

$$\underline{n} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{vmatrix} \underline{i} & 1 & 2 \\ \underline{j} & 4 & 3 \\ \underline{k} & 4 & 1 \end{vmatrix} = \begin{pmatrix} -8 \\ 7 \\ -5 \end{pmatrix}$$

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}\right).\begin{pmatrix} -8 \\ 7 \\ -5 \end{pmatrix} = 0 \Rightarrow -8x + 7(y+2) - 5(z+1) = 0$$

$\Rightarrow -8x + 7y - 5z = -9$ or $8x - 7y + 5z = 9$

Alternative version (once $\underline{n}$ has been found)

Let plane be $-8x + 7y - 5z = p$

As $\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$ lies in the plane, $-8(0) + 7(-2) - 5(-1) = p$;

so $p = -9$ etc

Alternative method

Eliminate s & t from the 3 simultaneous equations.

(5*) Lines and planes**

Find the line that is the reflection of the line $\frac{x-2}{3} = \frac{y}{4} = \frac{z+1}{1}$ in the plane $x - 2y + z = 4$

Solution

Let the intersection of the line and the plane be P, and suppose that Q is some other point on the line. Then we can find the reflection of Q in the plane (Q' say), by dropping a perpendicular from Q onto the plane, and then the required line will pass through P and Q'.

Writing the equation of the line as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ and substituting into the equation of the plane:

$(2 + 3\lambda) - 2(4\lambda) + (-1 + \lambda) = 4 \Rightarrow -4\lambda = 3;\ \lambda = -\frac{3}{4}$

so that P is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$

Setting $\lambda = 1$ (say), we can take Q to be $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

Now consider the perpendicular line dropped from Q onto the plane. Its direction vector is that of the normal to the plane, and so it has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let R be the point where the perpendicular line intersects the plane. Substituting into the equation of the plane gives:

$$(5 + \lambda) - 2(4 - 2\lambda) + (\lambda) = 4 \Rightarrow 6\lambda = 7;\ \lambda = \frac{7}{6}$$

So R is $\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \frac{7}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, and Q' will be $\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + 2\left(\frac{7}{6}\right)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{22}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix}$

Then , as P is $\begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$, the equation of the reflected line will be:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix} + \lambda \left[\begin{pmatrix} \frac{22}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix} - \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix} \right] = \frac{1}{12}\begin{pmatrix} -3 + \lambda(88 + 3) \\ -36 + \lambda(-8 + 36) \\ -21 + \lambda(28 + 21) \end{pmatrix}$$

$$\frac{1}{12}\begin{pmatrix} -3 + 91\lambda \\ -36 + 28\lambda \\ -21 + 49\lambda \end{pmatrix}$$

or, in cartesian form: $\frac{x+\frac{3}{12}}{91}=\frac{y+\frac{36}{12}}{28}=\frac{z+\frac{21}{12}}{49}$ or $\frac{x+\frac{3}{12}}{13}=\frac{y+\frac{36}{12}}{4}=\frac{z+\frac{21}{12}}{7}$

(6) Lines and planes**

(i)(a) Find the acute angle between the line $\frac{x}{2}=\frac{y+1}{-3}=\frac{z-2}{1}$ and the plane $x+y-2z=5$

(b) Show that the same angle is obtained if the line is written in the form

$\frac{x}{-2}=\frac{y+1}{3}=\frac{z-2}{-1}$ (ie without rearranging into the form in (a))

(ii)(a) Find the acute angle between the planes $x+4y-3z=7$

and $x-y+4z=2$

(b) Find the acute angle between the planes $x+4y-3z=7$ and $-x+y-4z=2$ (again, without rearranging the equation)

Solution

(i)(a) The angle between the line and the normal to the plane is given by

$\begin{pmatrix}\mathbf{2}\\\mathbf{-3}\\\mathbf{1}\end{pmatrix}.\begin{pmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{-2}\end{pmatrix}=\sqrt{\mathbf{14}}\sqrt{\mathbf{6}}\,\boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta}=\frac{\mathbf{-3}}{\sqrt{\mathbf{14}}\sqrt{\mathbf{6}}}=$ $\mathbf{-0.32733}$

and $\boldsymbol{\theta}=\mathbf{109.107}°$

The acute angle between these vectors is then $\mathbf{180-109.107=70.893}°$

The acute angle between the line and plane is then

$\mathbf{90 - 70.893 = 19.1}°$ (1dp)

(b) $\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \sqrt{14}\sqrt{6}\, \boldsymbol{cos\theta} \Rightarrow \boldsymbol{cos\theta} = \frac{3}{\sqrt{14}\sqrt{6}} = \mathbf{0.32733}$

and $\boldsymbol{\theta} = \mathbf{70.893}°$

As we have already found the acute angle between the line and the normal, the acute angle between the line and the plane is $\mathbf{90 - 70.893 = 19.1}°$ (1dp)

(ii) The angle between the normals to the planes is given by

$\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \sqrt{26}\sqrt{18}\, \boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{-15}{\sqrt{26}\sqrt{18}} = \mathbf{-0.69338}$

and $\boldsymbol{\theta} = \mathbf{133.898}°$

The acute angle between the planes themselves is $\mathbf{180 - 133.898 = 46.1}°$

(ii)(b) The angle between the normals to the planes is given by

$\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} . \begin{pmatrix} -1 \\ 1 \\ -4 \end{pmatrix} = \sqrt{26}\sqrt{18}\, \boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{15}{\sqrt{26}\sqrt{18}} = \mathbf{0.69338}$

and $\boldsymbol{\theta} = \mathbf{46.1}°$

The acute angle between the planes is also $\mathbf{46.1}°$.

(7***) Lines and planes

Find the line that is the reflection of the line $\frac{x-2}{3}=\frac{y}{4}=\frac{z+1}{1}$ in the plane $x-2y+z=4$

Solution

Let the intersection of the line and the plane be P, and suppose that Q is some other point on the line. Then we can find the reflection of Q in the plane (Q' say), by dropping a perpendicular from Q onto the plane, and then the required line will pass through P and Q'.

Writing the equation of the line as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ and substituting into the equation of the plane:

$$(2+3\lambda)-2(4\lambda)+(-1+\lambda)=4 \Rightarrow -4\lambda=3;\ \lambda=-\frac{3}{4}$$

so that P is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}-\frac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$

Setting $\lambda=1$ (say), we can take Q to be $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}+\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

Now consider the perpendicular line dropped from Q onto the plane. Its direction vector is that of the normal to the plane, and so it has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let R be the point where the perpendicular line intersects the plane. Substituting into the equation of the plane gives:

$$(5 + \lambda) - 2(4 - 2\lambda) + (\lambda) = 4 \Rightarrow 6\lambda = 7;\ \lambda = \frac{7}{6}$$

So R is $\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \frac{7}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, and Q' will be $\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + 2\left(\frac{7}{6}\right)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{22}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix}$

Then , as P is $\begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$, the equation of the reflected line will be:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix} + \lambda\left[\begin{pmatrix} \frac{22}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix} - \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}\right] = \frac{1}{12}\begin{pmatrix} -3 + \lambda(88 + 3) \\ -36 + \lambda(-8 + 36) \\ -21 + \lambda(28 + 21) \end{pmatrix}$$

$$\frac{1}{12}\begin{pmatrix} -3 + 91\lambda \\ -36 + 28\lambda \\ -21 + 49\lambda \end{pmatrix}$$

or, in cartesian form: $\frac{x + \frac{3}{12}}{91} = \frac{y + \frac{36}{12}}{28} = \frac{z + \frac{21}{12}}{49}$ or $\frac{x + \frac{1}{4}}{13} = \frac{y + 3}{4} = \frac{z + \frac{7}{4}}{7}$

(8***) Lines

Find the distance between the lines $\frac{x+1}{1} = \frac{y+2}{2}; z = 4$ and $\frac{x+3}{1} = \frac{y-6}{2}$; $z = 7$, leaving your answer in exact form.

Solution

Method 1

The lines are parallel.

Choose a point on one of the lines; eg $P = (-3, 6, 7)$ on the 2nd line.

To find the distance of this point from the 1st line:

A general point, Q on the 1st line is $\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} -1 + \lambda \\ -2 + 2\lambda \\ 4 \end{pmatrix}$

Then $\overrightarrow{PQ} = \begin{pmatrix} -1 + \lambda \\ -2 + 2\lambda \\ 4 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 + \lambda \\ -8 + 2\lambda \\ -3 \end{pmatrix}$

We want $\overrightarrow{PQ}$ to be perpendicular to the 1st line,

so that $\begin{pmatrix} 2 + \lambda \\ -8 + 2\lambda \\ -3 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0$

$\Rightarrow 2 + \lambda - 16 + 4\lambda = 0 \;\; \Rightarrow 5\lambda = 14; \lambda = \frac{14}{5}$

Then $\overrightarrow{\boldsymbol{PQ}} = \begin{pmatrix} \frac{24}{5} \\ -\frac{12}{5} \\ -\frac{15}{5} \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 8 \\ -4 \\ -5 \end{pmatrix}$ and the required distance is

$\frac{3}{5}\sqrt{64 + 16 + 25}$

$= \frac{3\sqrt{105}}{5}$

Method 2

Choose a point on each line; eg $\boldsymbol{R} = (-1, -2, 4)$ on the 1st line, and
$\boldsymbol{P} = (-3, 6, 7)$ on the 2nd line.

Then $\overrightarrow{\boldsymbol{PR}} = \begin{pmatrix} 2 \\ -8 \\ -3 \end{pmatrix}$ and the required distance is $\left| \frac{\begin{pmatrix} 2 \\ -8 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right|} \right|$

$= \left| \frac{\begin{vmatrix} \underline{i} & 2 & 1 \\ \underline{j} & -8 & 2 \\ \underline{k} & -3 & 0 \end{vmatrix}}{\sqrt{5}} \right| = \frac{1}{\sqrt{5}}\left|\begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix}\right| = \frac{3}{\sqrt{5}}\left|\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}\right| = \frac{3}{\sqrt{5}}\sqrt{21} = \frac{3\sqrt{105}}{5}$

(9*) Lines**

(i) Show the lines $\frac{x-1}{2} = \frac{y+3}{5} = \frac{z-2}{3}$ and $\frac{x}{1} = \frac{y-4}{2} = \frac{z+1}{2}$ are skew.

(ii) Find the shortest distance between the lines and identify the points on the lines at which this shortest distance occurs.

Solution

(i) The lines can be rewritten in parametric form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ -3+5\lambda \\ 2+3\lambda \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mu \\ 4+2\mu \\ -1+2\mu \end{pmatrix}$$

A point of intersection would then satisfy

$\mathbf{1 + 2\lambda = \mu}$ (1)
$\mathbf{-3 + 5\lambda = 4 + 2\mu}$ (2)
$\mathbf{2 + 3\lambda = -1 + 2\mu}$ (3)

Substituting from (1) into (2) & (3) gives:

$\mathbf{-3 + 5\lambda = 4 + 2(1 + 2\lambda)}$ or $\mathbf{-9 = -\lambda}$, so that $\boldsymbol{\lambda} = \mathbf{9}$

and $\mathbf{2 + 3\lambda = -1 + 2(1 + 2\lambda)}$ or $\mathbf{1 = \lambda}$,

and so there is no point of intersection.

Also, the direction vectors of the lines are not parallel, and so the lines are skew.

(ii) [There are various methods for finding the shortest distance, but not all of them find the points on the lines where the shortest distance occurs. The first method given below is relatively straightforward, and doesn't involve the vector product.]

Method 1

From (i), general points on the two lines are

$$\overrightarrow{OX} = \begin{pmatrix} 1+2\lambda \\ -3+5\lambda \\ 2+3\lambda \end{pmatrix} \text{ and } \overrightarrow{OY} = \begin{pmatrix} \mu \\ 4+2\mu \\ -1+2\mu \end{pmatrix}$$

At the closest approach of the two lines, $\overrightarrow{XY}$ will be perpendicular to both lines, so that

$\overrightarrow{\boldsymbol{XY}}.\begin{pmatrix}\mathbf{2}\\\mathbf{5}\\\mathbf{3}\end{pmatrix}=\mathbf{0}$ and $\overrightarrow{\boldsymbol{XY}}.\begin{pmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{2}\end{pmatrix}=\mathbf{0}$, so that

$$\begin{pmatrix}\boldsymbol{\mu}-(\mathbf{1}+\mathbf{2}\boldsymbol{\lambda})\\\mathbf{4}+\mathbf{2}\boldsymbol{\mu}-(-\mathbf{3}+\mathbf{5}\boldsymbol{\lambda})\\-\mathbf{1}+\mathbf{2}\boldsymbol{\mu}-(\mathbf{2}+\mathbf{3}\boldsymbol{\lambda})\end{pmatrix}.\begin{pmatrix}\mathbf{2}\\\mathbf{5}\\\mathbf{3}\end{pmatrix}=\mathbf{0}\text{ and}$$

$$\begin{pmatrix}\boldsymbol{\mu}-(\mathbf{1}+\mathbf{2}\boldsymbol{\lambda})\\\mathbf{4}+\mathbf{2}\boldsymbol{\mu}-(-\mathbf{3}+\mathbf{5}\boldsymbol{\lambda})\\-\mathbf{1}+\mathbf{2}\boldsymbol{\mu}-(\mathbf{2}+\mathbf{3}\boldsymbol{\lambda})\end{pmatrix}.\begin{pmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{2}\end{pmatrix}=\mathbf{0},$$

giving $(\mathbf{2}\boldsymbol{\mu}-\mathbf{2}-\mathbf{4}\boldsymbol{\lambda})+(\mathbf{35}+\mathbf{10}\boldsymbol{\mu}-\mathbf{25}\boldsymbol{\lambda})+(-\mathbf{9}+\mathbf{6}\boldsymbol{\mu}-\mathbf{9}\boldsymbol{\lambda})=\mathbf{0}$

or $\mathbf{18}\boldsymbol{\mu}-\mathbf{38}\boldsymbol{\lambda}=-\mathbf{24}$; ie $\mathbf{9}\boldsymbol{\mu}-\mathbf{19}\boldsymbol{\lambda}=-\mathbf{12}$ (1)

and $(\boldsymbol{\mu}-\mathbf{1}-\mathbf{2}\boldsymbol{\lambda})+(\mathbf{14}+\mathbf{4}\boldsymbol{\mu}-\mathbf{10}\boldsymbol{\lambda})+(-\mathbf{6}+\mathbf{4}\boldsymbol{\mu}-\mathbf{6}\boldsymbol{\lambda})=\mathbf{0}$

$\mathbf{9}\boldsymbol{\mu}-\mathbf{18}\boldsymbol{\lambda}=-\mathbf{7}$ (2)

Then $(\mathbf{1})-(\mathbf{2})\Rightarrow-\boldsymbol{\lambda}=-\mathbf{5}$, so that $\boldsymbol{\lambda}=\mathbf{5}$ and, from (2),

$\boldsymbol{\mu}=\frac{\mathbf{1}}{\mathbf{9}}(\mathbf{18}(\mathbf{5})-\mathbf{7})=\frac{\mathbf{83}}{\mathbf{9}}$

So $\overrightarrow{\boldsymbol{OX}}=\begin{pmatrix}\mathbf{11}\\\mathbf{22}\\\mathbf{17}\end{pmatrix}$ and $\overrightarrow{\boldsymbol{OY}}=\frac{\mathbf{1}}{\mathbf{9}}\begin{pmatrix}\mathbf{83}\\\mathbf{202}\\\mathbf{157}\end{pmatrix}$

and $\overrightarrow{\boldsymbol{XY}}=\frac{\mathbf{1}}{\mathbf{9}}\begin{pmatrix}\mathbf{83}-\mathbf{99}\\\mathbf{202}-\mathbf{198}\\\mathbf{157}-\mathbf{153}\end{pmatrix}=\frac{\mathbf{1}}{\mathbf{9}}\begin{pmatrix}-\mathbf{16}\\\mathbf{4}\\\mathbf{4}\end{pmatrix}=\frac{\mathbf{4}}{\mathbf{9}}\begin{pmatrix}-\mathbf{4}\\\mathbf{1}\\\mathbf{1}\end{pmatrix}$,

so that $|\overrightarrow{\boldsymbol{XY}}|=\frac{\mathbf{4}}{\mathbf{9}}\sqrt{\mathbf{16}+\mathbf{1}+\mathbf{1}}=\frac{\mathbf{4}\sqrt{\mathbf{18}}}{\mathbf{9}}=\frac{\mathbf{4}\sqrt{\mathbf{2}}}{\mathbf{3}}$

Method 2 (using the vector product)

If $\underline{\hat{\boldsymbol{n}}}$ is a unit vector perpendicular to both lines, then we need $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ such that $\overrightarrow{\boldsymbol{OX}} + \boldsymbol{d}\underline{\hat{\boldsymbol{n}}} = \overrightarrow{\boldsymbol{OY}}$, and the shortest distance will then be $|\boldsymbol{d}|$.

A vector perpendicular to both lines is $\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{vmatrix} \underline{\boldsymbol{i}} & \boldsymbol{2} & \boldsymbol{1} \\ \underline{\boldsymbol{j}} & \boldsymbol{5} & \boldsymbol{2} \\ \underline{\boldsymbol{k}} & \boldsymbol{3} & \boldsymbol{2} \end{vmatrix}$

$= \begin{pmatrix} \boldsymbol{4} \\ \boldsymbol{-1} \\ \boldsymbol{-1} \end{pmatrix}$, so that $\underline{\hat{\boldsymbol{n}}} = \frac{\boldsymbol{1}}{\sqrt{\boldsymbol{18}}}\begin{pmatrix} \boldsymbol{4} \\ \boldsymbol{-1} \\ \boldsymbol{-1} \end{pmatrix}$

Then $\overrightarrow{\boldsymbol{OX}} + \boldsymbol{d}\underline{\hat{\boldsymbol{n}}} = \overrightarrow{\boldsymbol{OY}}$ gives $\begin{pmatrix} \boldsymbol{1+2\lambda} \\ \boldsymbol{-3+5\lambda} \\ \boldsymbol{2+3\lambda} \end{pmatrix} + \boldsymbol{D}\begin{pmatrix} \boldsymbol{4} \\ \boldsymbol{-1} \\ \boldsymbol{-1} \end{pmatrix} =$
$\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{4+2\mu} \\ \boldsymbol{-1+2\mu} \end{pmatrix}$,
where $\boldsymbol{D} = \frac{\boldsymbol{d}}{\sqrt{\boldsymbol{18}}}$,

so that $\boldsymbol{2\lambda + 4D - \mu = -1}$ (1)
$\boldsymbol{5\lambda - D - 2\mu = 7}$ (2)
$3\boldsymbol{\lambda - D - 2\mu = -3}$ (3)

Then $(\boldsymbol{2}) - (\boldsymbol{3}) \Rightarrow \boldsymbol{2\lambda = 10}$, so that $\boldsymbol{\lambda = 5}$

and (1) & (2) become $\boldsymbol{4D - \mu = -11}$ (4) and $\boldsymbol{-D - 2\mu = -18}$ (5)

Then $\boldsymbol{2}(\boldsymbol{4}) - (\boldsymbol{5}) \Rightarrow \boldsymbol{9D = -4}$, so that $|\boldsymbol{d}| = \sqrt{\boldsymbol{18}}|\boldsymbol{D}| = \frac{\boldsymbol{4\sqrt{18}}}{\boldsymbol{9}} = \frac{\boldsymbol{4\sqrt{2}}}{\boldsymbol{3}}$

and, from (1), $\boldsymbol{\mu = 10 - \frac{16}{9} + 1 = \frac{83}{9}}$

and $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ can then be found, as in (i).

(10**) Lines

Given that $A = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $B = \begin{pmatrix}-4\\3\\1\end{pmatrix}$, $C = \begin{pmatrix}0\\1\\2\end{pmatrix}$, $D = \begin{pmatrix}p\\4\\-4\end{pmatrix}$

(i) Write down the equations of the lines AB and CD (both extended)

(ii) Find $\overrightarrow{AB} \times \overrightarrow{CD}$

(iii) For what value of p are the lines AB and CD parallel? (2 methods)

Solution

(i) Write down the equations of the lines AB and CD (both extended)

$$\underline{r_{AB}} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-4-1\\3-2\\1-3\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-5\\1\\-2\end{pmatrix}$$

and $\underline{r_{CD}} = \begin{pmatrix}0\\1\\2\end{pmatrix} + \mu\begin{pmatrix}p\\4-1\\-4-2\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix} + \mu\begin{pmatrix}p\\3\\-6\end{pmatrix}$

Note

or eg $\underline{r_{AB}} = \begin{pmatrix}-4\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}5\\-1\\2\end{pmatrix}$

(ii) Find $\overrightarrow{AB} \times \overrightarrow{CD}$

$$\begin{pmatrix} \underline{i} & -5 & p \\ \underline{j} & 1 & 3 \\ \underline{k} & -2 & -6 \end{pmatrix} = -(30+2p)\underline{j} - (15+p)\underline{k} = -(15+p)(2\underline{j}+\underline{k})$$

(iii) For what value of p are the lines AB and CD parallel? (2 methods)

Method 1: Direction vectors $\begin{pmatrix} -5 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} p \\ 3 \\ -6 \end{pmatrix}$ need to be parallel; hence $p = -15$

Method 2: $\overrightarrow{AB} \times \overrightarrow{CD}$ must be zero

Hence $15 + p = 0$

(11) Planes**

Find the plane containing the points $(2,-1,4), (-3,4,2)$ and $(1,0,5)$, in Cartesian form

Solution

Method 1

In parametric form, it is:

$$\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda\left[\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}\right] + \mu[\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}]$$

or $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$x = 2 - 5\lambda - \mu$ (1)

$y = -1 + 5\lambda + \mu$ (2)

$z = 4 - 2\lambda + \mu$ (3)

Eliminating λ & μ: $(1) + (2) \Rightarrow x + y = 1$

Method 2

The normal to the plane is perpendicular to both

$\begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$; eg $\begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{vmatrix} \underline{i} & -5 & -1 \\ \underline{j} & 5 & 1 \\ \underline{k} & -2 & 1 \end{vmatrix}$

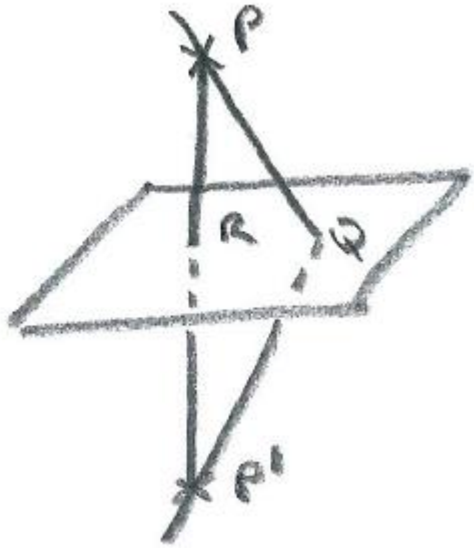
$= \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$,

so that eq'n is $\underline{r}.\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}.\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ or $x + y = 1$

(12*) Lines and planes**

Find the reflection of the line $\frac{x-2}{3} = \frac{y+4}{1}$; $z = 3$ in the plane $y = 4$

Solution



Let P be $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$, say.

Q is intersection of the line and plane :

Line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

Substituting into the eq'n of the plane: $-4 + \lambda = 4 \Rightarrow \lambda = 8$

So Q is $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + 8 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 26 \\ 4 \\ 3 \end{pmatrix}$

Line PR is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

R is intersection of PR and the plane:

$-4 + \mu = 4 \Rightarrow \mu = 8$

So P' is $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + 2(8) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix}$

Eq'n of P'Q is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta[\begin{pmatrix} 26 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix}]$

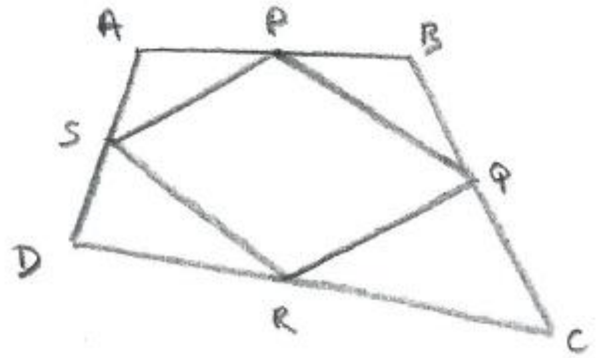
ie $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta \begin{pmatrix} 24 \\ -8 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta' \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

or $\frac{x-2}{3} = \frac{y-12}{-1}; z = 3$

(13***) Problem

Use vectors to prove that the mid-points of the sides of any quadrilateral form the vertices of a parallelogram.

Solution



Referring to the diagram (where $\underline{a} = \overrightarrow{OA}\ \ etc$),

$\underline{q} - \underline{p} = \frac{1}{2}(\underline{b} + \underline{c}) - \frac{1}{2}(\underline{a} + \underline{b}) = \frac{1}{2}(\underline{c} - \underline{a})$

and $\underline{r} - \underline{s} = \frac{1}{2}(\underline{c} + \underline{d}) - \frac{1}{2}(\underline{a} + \underline{d}) = \frac{1}{2}(\underline{c} - \underline{a}) = \underline{q} - \underline{p}$

So the sides PQ & SR are of equal length and parallel.

This means that $PQRS$ is a parallelogram.

(14) Distance from point to plane**

(i) Find the intersection of the line $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and the plane $3x + y + 4z = 77$

(ii) Find the shortest distance from the point $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ to the plane

$3x + y + 4z = 77$

Solution

(i) For a point on the line, $x = 2 + 3\lambda, y = -1 + \lambda, z = 5 + 4\lambda$

Substituting into the eq'n of the plane:

$3(2 + 3\lambda) + (-1 + \lambda) + 4(5 + 4\lambda) = 77$

$\Rightarrow 26\lambda = 77 - 25 \Rightarrow \lambda = \frac{52}{26} = 2$

$\Rightarrow$ point of intersection has position vector $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 1 \\ 13 \end{pmatrix}$

(ii) From (i), nearest point on the plane is $\begin{pmatrix} 8 \\ 1 \\ 13 \end{pmatrix}$,

so that shortest distance is $\sqrt{(8-2)^2 + (1-[-1])^2 + (13-5)^2}$

$= \sqrt{36 + 4 + 64} = \sqrt{104} = 2\sqrt{26}$

Alternative method

From (i), shortest distance is $|\lambda||\underline{n}|$, where λ corresponds to the point of intersection of the line and plane, and $\underline{n}$ is the normal vector for the plane; ie $2\sqrt{3^2+1^2+4^2} = 2\sqrt{26}$

(15***) Distance from point to plane

Show that the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$

Solution

$(\underline{p} + \lambda\underline{n}).\underline{n} = d \Rightarrow \underline{p}.\underline{n} + \lambda|\underline{n}|^2 = d$

$\Rightarrow \lambda = \frac{d-\underline{p}.\underline{n}}{|\underline{n}|^2}$

So shortest distance $= |\lambda||\underline{n}| = \frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$

(16**) Distance from point to plane

(i) Given that the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$, what is the significance of $\frac{d}{|\underline{n}|}$ if $d > 0$?

(ii) Find the equation of the plane that is parallel to $\underline{r}.\underline{n} = d$ and contains the point $\underline{p}$.

(iii) Hence deduce the formula for the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$

Solution

(i) $\frac{d}{|\underline{n}|}$ is the distance of the plane $\underline{r}.\underline{n} = d$ from the Origin,

when $d > 0$

(ii) $\underline{r}.\underline{n} = \underline{p}.\underline{n}$

(iii) The shortest distance from the plane $\underline{r}.\underline{n} = d$ to the Origin is $\frac{d}{|\underline{n}|}$.

The plane parallel to $\underline{r}.\underline{n} = d$, containing $\underline{p}$ has equation

$\underline{r}.\underline{n} = \underline{p}.\underline{n}$, and its shortest distance from the Origin is $\frac{\underline{p}.\underline{n}}{|\underline{n}|}$

Hence the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$

is $\left|\frac{\underline{p}.\underline{n}}{|\underline{n}|} - \frac{d}{|\underline{n}|}\right| = \frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$

(17) Volume of tetrahedron**

Find the volume of the tetrahedron with corners
$(2,1,3), (-1,5,0), (4,4,7), (8,2,2)$

Method 1

Label the corners as follows:

$A(2,1,3), B(-1,5,0), C(4,4,7), D(8,2,2)$

Then volume $= \frac{1}{3}.\frac{1}{2}|\overrightarrow{AB}.(\overrightarrow{AC} \times \overrightarrow{AD})|$

(based on $\frac{1}{3}$ × area of triangle ABC × perpendicular height)

and $\overrightarrow{AB}.(\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} -3 & 2 & 6 \\ 4 & 3 & 1 \\ -3 & 4 & -1 \end{vmatrix}$

$= -3(-7) - 4(-26) - 3(-16) = 21 + 104 + 48 = 173$

So volume is $\frac{173}{6}$ units^3.

Method 2a (much longer, but good practice!)

Volume $= \frac{1}{3} \times$ area of base ABC $\times$ perpendicular height

Area of base ABC $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

and $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & -3 & 2 \\ \underline{j} & 4 & 3 \\ \underline{k} & -3 & 4 \end{vmatrix} = \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$,

so that Area of base ABC $= \frac{1}{2}\sqrt{25^2 + 6^2 + (-17)^2} = \frac{5}{2}\sqrt{38}$

The perpendicular height is the shortest distance from D to the plane ABC.

A normal to the plane ABC is $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$ (already calculated).

And the equation of the plane ABC is

$$\underline{r}.\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = 50 + 6 - 51 = 5\,,$$

taking $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ as a point in the plane.

Let the point of intersection of the perpendicular from D onto the plane ABC be P, given by the following point on the perpendicular:

$$\begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$$

As P lies in the plane ABC,

$$\left(\begin{pmatrix}8\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}25\\6\\-17\end{pmatrix}\right).\begin{pmatrix}25\\6\\-17\end{pmatrix}=5$$

Then $178+950\lambda=5$, so that $\lambda=-\frac{173}{950}$

and the perpendicular height is $|\lambda|\left|\begin{matrix}25\\6\\-17\end{matrix}\right|$

$=\frac{173}{950}\sqrt{25^2+6^2+(-17)^2}=\frac{173}{950}\,.5\sqrt{38}=\frac{173}{190}\sqrt{38}$

Hence the volume of the tetrahedron is

$\frac{1}{3}.\frac{5}{2}\sqrt{38}\,.\frac{173}{190}\sqrt{38}=\frac{173}{6}$ units^3

Method 2b (even longer)

As Method 2a, but determining λ as follows:

For P to be a point in the plane ABC,

$$\begin{pmatrix}8\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}25\\6\\-17\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}-3\\4\\-3\end{pmatrix}+\theta\begin{pmatrix}2\\3\\4\end{pmatrix},$$

as $\overrightarrow{OA}=\begin{pmatrix}2\\1\\3\end{pmatrix}$ is a point in the plane, and $\overrightarrow{AB}=\begin{pmatrix}-3\\4\\-3\end{pmatrix}$ and

$\overrightarrow{AC}=\begin{pmatrix}2\\3\\4\end{pmatrix}$ are directions parallel to the plane

Then $\begin{pmatrix} 25 & 3 & -2 \\ 6 & -4 & -3 \\ -17 & 3 & -4 \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \\ \theta \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \\ 1 \end{pmatrix}$

$$\begin{vmatrix} 25 & 3 & -2 \\ 6 & -4 & -3 \\ -17 & 3 & -4 \end{vmatrix} = 25(25) - 6(-6) - 17(-17) = 950$$

$$\begin{pmatrix} 25 & 3 & -2 \\ 6 & -4 & -3 \\ -17 & 3 & -4 \end{pmatrix}^{-1} = \frac{1}{950}\begin{pmatrix} 25 & 75 & -50 \\ 6 & -134 & -126 \\ -17 & 63 & -118 \end{pmatrix}^{T}$$

So $\begin{pmatrix} \lambda \\ \mu \\ \theta \end{pmatrix} = \frac{1}{950}\begin{pmatrix} 25 & 6 & -17 \\ 75 & -134 & 63 \\ -50 & -126 & -118 \end{pmatrix}\begin{pmatrix} -6 \\ -1 \\ 1 \end{pmatrix}$

$= \frac{1}{950}\begin{pmatrix} -173 \\ -253 \\ 308 \end{pmatrix}$,

so that $\lambda = -\frac{173}{950}$