Vectors Exercises - Moderate (Sol'ns) (15 pages; 21/1/21)

(1) Scalar product

Show that if $|\underline{a}-\underline{b}| = |\underline{a}+\underline{b}|$, then $\underline{a}$ & $\underline{b}$ are perpendicular.

Solution

$|\underline{a}-\underline{b}| = |\underline{a}+\underline{b}| \Rightarrow |\underline{a}-\underline{b}|^2 = |\underline{a}+\underline{b}|^2$

$\Rightarrow (\underline{a}-\underline{b}).(\underline{a}-\underline{b}) = (\underline{a}+\underline{b}).(\underline{a}+\underline{b})$

$[\ \underline{x}.\underline{x} = |\underline{x}|\,|\underline{x}|cos0° = |\underline{x}|^2\ ]$

$\Rightarrow \underline{a}.\underline{a} - \underline{a}.\underline{b} - \underline{b}.\underline{a} + \underline{b}.\underline{b} = \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{b}.\underline{a} + \underline{b}.\underline{b}$

$\Rightarrow -2\underline{a}.\underline{b} = 2\underline{a}.\underline{b}$ [since $\underline{a}.\underline{b} = \underline{b}.\underline{a}$]

$\Rightarrow \underline{a}.\underline{b} = 0$

and hence $\underline{a}$ & $\underline{b}$ are perpendicular

[Geometrically, $|\underline{a}-\underline{b}|$&$|\underline{a}+\underline{b}|$ are the 'short' and 'long' diagonals of the parallelogram formed from the adjacent sides $\underline{a}$ & $\underline{b}$. When these diagonals are equal, the parallelogram is a rectangle.]

(2) Planes

Find the cartesian form of the plane

$$\underline{r} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Solution

$$\underline{n} = \begin{pmatrix}1\\4\\4\end{pmatrix} \times \begin{pmatrix}2\\3\\1\end{pmatrix} = \begin{vmatrix}\underline{i} & 1 & 2\\ \underline{j} & 4 & 3\\ \underline{k} & 4 & 1\end{vmatrix} = \begin{pmatrix}-8\\7\\-5\end{pmatrix}$$

$$\left(\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}0\\-2\\-1\end{pmatrix}\right).\begin{pmatrix}-8\\7\\-5\end{pmatrix} = 0 \Rightarrow -8x + 7(y+2) - 5(z+1) = 0$$

$\Rightarrow -8x + 7y - 5z = -9$ or $8x - 7y + 5z = 9$

Alternative version (once $\underline{n}$ has been found)

Let plane be $-8x + 7y - 5z = p$

As $\begin{pmatrix}0\\-2\\-1\end{pmatrix}$ lies in the plane, $-8(0) + 7(-2) - 5(-1) = p$;

so $p = -9$ etc

Alternative method

Eliminate s & t from the 3 simultaneous equations.

(3) Lines and planes

(i)(a) Find the acute angle between the line $\boldsymbol{\frac{x}{2} = \frac{y+1}{-3} = \frac{z-2}{1}}$ and the plane $\boldsymbol{x + y - 2z = 5}$

(b) Show that the same angle is obtained if the line is written in the form

$\boldsymbol{\frac{x}{-2} = \frac{y+1}{3} = \frac{z-2}{-1}}$ (ie without rearranging into the form in (a))

(ii)(a) Find the acute angle between the planes $\boldsymbol{x + 4y - 3z = 7}$

and $\boldsymbol{x - y + 4z = 2}$

(b) Find the acute angle between the planes $\boldsymbol{x + 4y - 3z = 7}$ and
$\boldsymbol{-x + y - 4z = 2}$ (again, without rearranging the equation)

Solution

(i)(a) The angle between the line and the normal to the plane is given by

$\begin{pmatrix} \mathbf{2} \\ \mathbf{-3} \\ \mathbf{1} \end{pmatrix} . \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-2} \end{pmatrix} = \sqrt{\mathbf{14}}\sqrt{\mathbf{6}}\, \boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{\mathbf{-3}}{\sqrt{\mathbf{14}}\sqrt{\mathbf{6}}} = \mathbf{-0.32733}$

and $\boldsymbol{\theta} = \mathbf{109.107}°$

The acute angle between these vectors is then $\mathbf{180 - 109.107 = 70.893}°$

The acute angle between the line and plane is then

$\mathbf{90 - 70.893 = 19.1}°$ (1dp)

(b) $\begin{pmatrix} \mathbf{-2} \\ \mathbf{3} \\ \mathbf{-1} \end{pmatrix} . \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-2} \end{pmatrix} = \sqrt{\mathbf{14}}\sqrt{\mathbf{6}}\, \boldsymbol{cos\theta} \Rightarrow \boldsymbol{cos\theta} = \frac{\mathbf{3}}{\sqrt{\mathbf{14}}\sqrt{\mathbf{6}}} = \mathbf{0.32733}$

and $\boldsymbol{\theta} = \mathbf{70.893}°$

As we have already found the acute angle between the line and the normal, the acute angle between the line and the plane is $\mathbf{90 - 70.893 = 19.1}°$ (1dp)

(ii) The angle between the normals to the planes is given by

$\begin{pmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{-3} \end{pmatrix} . \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{4} \end{pmatrix} = \sqrt{\mathbf{26}}\sqrt{\mathbf{18}}\ \boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{\mathbf{-15}}{\sqrt{\mathbf{26}}\sqrt{\mathbf{18}}} =$
$\mathbf{-0.69338}$

and $\boldsymbol{\theta} = \mathbf{133.898}°$

The acute angle between the planes themselves is $\mathbf{180} -$
$\mathbf{133.898} = \mathbf{46.1}°$

(ii)(b) The angle between the normals to the planes is given by

$\begin{pmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{-3} \end{pmatrix} . \begin{pmatrix} \mathbf{-1} \\ \mathbf{1} \\ \mathbf{-4} \end{pmatrix} = \sqrt{\mathbf{26}}\sqrt{\mathbf{18}}\ \boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{\mathbf{15}}{\sqrt{\mathbf{26}}\sqrt{\mathbf{18}}} =$
$\mathbf{0.69338}$

and $\boldsymbol{\theta} = \mathbf{46.1}°$

The acute angle between the planes is also $\mathbf{46.1}°$.

(4) Lines

Given that $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $B = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $D = \begin{pmatrix} p \\ 4 \\ -4 \end{pmatrix}$

(i) Write down the equations of the lines AB and CD (both extended)

(ii) Find $\overrightarrow{AB} \times \overrightarrow{CD}$

(iii) For what value of p are the lines AB and CD parallel? (2 methods)

Solution

(i) Write down the equations of the lines AB and CD (both extended)

$$\underline{r_{AB}} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-4-1\\3-2\\1-3\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-5\\1\\-2\end{pmatrix}$$

and $\underline{r_{CD}} = \begin{pmatrix}0\\1\\2\end{pmatrix} + \mu\begin{pmatrix}p\\4-1\\-4-2\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix} + \mu\begin{pmatrix}p\\3\\-6\end{pmatrix}$

Note

or eg $\underline{r_{AB}} = \begin{pmatrix}-4\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}5\\-1\\2\end{pmatrix}$

(ii) Find $\overrightarrow{AB} \times \overrightarrow{CD}$

$$\begin{pmatrix}\underline{i} & -5 & p\\ \underline{j} & 1 & 3\\ \underline{k} & -2 & -6\end{pmatrix} = -(30+2p)\underline{j} - (15+p)\underline{k} = -(15+p)(2\underline{j}+\underline{k})$$

(iii) For what value of p are the lines AB and CD parallel? (2 methods)

Method 1: Direction vectors $\begin{pmatrix}-5\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}p\\3\\-6\end{pmatrix}$ need to be parallel; hence $p = -15$

Method 2: $\overrightarrow{AB} \times \overrightarrow{CD}$ must be zero

Hence $15 + p = 0$

(5) Planes

Find the plane containing the points
$(2,-1,4), (-3,4,2)$ and $(1,0,5)$, in Cartesian form

Solution

Method 1

In parametric form, it is:

$$\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \left[\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right] + \mu [\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}]$$

or $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$x = 2 - 5\lambda - \mu \quad (1)$

$y = -1 + 5\lambda + \mu \quad (2)$

$z = 4 - 2\lambda + \mu \quad (3)$

Eliminating λ & μ: $(1) + (2) \Rightarrow x + y = 1$

Method 2

The normal to the plane is perpendicular to both

$\begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$; eg $\begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{vmatrix} \underline{i} & -5 & -1 \\ \underline{j} & 5 & 1 \\ \underline{k} & -2 & 1 \end{vmatrix}$

$= \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$,

so that eq'n is $\underline{r}.\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}.\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ or $x + y = 1$

(6) Distance from point to plane

(i) Find the intersection of the line $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and the plane $3x + y + 4z = 77$

(ii) Find the shortest distance from the point $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ to the plane $3x + y + 4z = 77$

Solution

(i) For a point on the line, $x = 2 + 3\lambda, y = -1 + \lambda, z = 5 + 4\lambda$

Substituting into the eq'n of the plane:

$3(2 + 3\lambda) + (-1 + \lambda) + 4(5 + 4\lambda) = 77$

$\Rightarrow 26\lambda = 77 - 25 \Rightarrow \lambda = \frac{52}{26} = 2$

$\Rightarrow$ point of intersection has position vector $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 1 \\ 13 \end{pmatrix}$

(ii) From (i), nearest point on the plane is $\begin{pmatrix} 8 \\ 1 \\ 13 \end{pmatrix}$, so that shortest distance is $\sqrt{(8-2)^2+(1-[-1])^2+(13-5)^2}$

$=\sqrt{36+4+64}=\sqrt{104}=2\sqrt{26}$

Alternative method

From (i), shortest distance is $|\lambda||\underline{n}|$, where λ corresponds to the point of intersection of the line and plane, and $\underline{n}$ is the normal vector for the plane; ie $2\sqrt{3^2+1^2+4^2}=2\sqrt{26}$

(7) Distance from point to plane

(i) Given that the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n}=d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$, what is the significance of $\frac{d}{|\underline{n}|}$ if $d>0$?

(ii) Find the equation of the plane that is parallel to $\underline{r}.\underline{n}=d$ and contains the point $\underline{p}$.

(iii) Hence deduce the formula for the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n}=d$

Solution

(i) $\frac{d}{|\underline{n}|}$ is the distance of the plane $\underline{r}.\underline{n}=d$ from the Origin,

when $d>0$

(ii) $\underline{r}.\underline{n}=\underline{p}.\underline{n}$

(iii) The shortest distance from the plane $\underline{r}.\underline{n}=d$ to the Origin is $\frac{d}{|\underline{n}|}$.

The plane parallel to $\underline{r}.\underline{n}=d$, containing $\underline{p}$ has equation

$\underline{r}.\underline{n} = \underline{p}.\underline{n}$, and its shortest distance from the Origin is $\frac{\underline{p}.\underline{n}}{|\underline{n}|}$

Hence the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$

is $\left|\frac{\underline{p}.\underline{n}}{|\underline{n}|} - \frac{d}{|\underline{n}|}\right| = \frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$

(8) Volume of tetrahedron

Find the volume of the tetrahedron with corners
$(2, 1, 3), (-1, 5, 0), (4, 4, 7), (8, 2, 2)$

Solution

Method 1

Label the corners as follows:

$A(2, 1, 3), B(-1, 5, 0), C(4, 4, 7), D(8, 2, 2)$

Then volume $= \frac{1}{3} . \frac{1}{2} |\overrightarrow{AB}.(\overrightarrow{AC} \times \overrightarrow{AD})|$

(based on $\frac{1}{3}$ × area of triangle ABC × perpendicular height)

and $\overrightarrow{AB}.(\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} -3 & 2 & 6 \\ 4 & 3 & 1 \\ -3 & 4 & -1 \end{vmatrix}$

$= -3(-7) - 4(-26) - 3(-16) = 21 + 104 + 48 = 173$

So volume is $\frac{173}{6}$ units^3.

Method 2a (much longer, but good practice!)

Volume $= \frac{1}{3}$ × area of base ABC × perpendicular height

Area of base ABC $= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$

and $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & -3 & 2 \\ \underline{j} & 4 & 3 \\ \underline{k} & -3 & 4 \end{vmatrix} = \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$,

so that Area of base ABC $= \frac{1}{2}\sqrt{25^2 + 6^2 + (-17)^2} = \frac{5}{2}\sqrt{38}$

The perpendicular height is the shortest distance from D to the plane ABC.

A normal to the plane ABC is $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$ (already calculated).

And the equation of the plane ABC is

$$\underline{r}.\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = 50 + 6 - 51 = 5\,,$$

taking $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ as a point in the plane.

Let the point of intersection of the perpendicular from D onto the plane ABC be P, given by the following point on the perpendicular:

$$\begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$$

As P lies in the plane ABC,

$$\left(\begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}\right).\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = 5$$

Then $178 + 950\lambda = 5$, so that $\lambda = -\frac{173}{950}$

and the perpendicular height is $|\lambda|\begin{vmatrix}25\\6\\-17\end{vmatrix}$

$= \frac{173}{950}\sqrt{25^2+6^2+(-17)^2} = \frac{173}{950}\ .5\sqrt{38} = \frac{173}{190}\sqrt{38}$

Hence the volume of the tetrahedron is

$\frac{1}{3}.\frac{5}{2}\sqrt{38}\,.\frac{173}{190}\sqrt{38} = \frac{173}{6}$ units^3

Method 2b (even longer)

As Method 2a, but determining λ as follows:

For P to be a point in the plane ABC,

$$\begin{pmatrix}8\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}25\\6\\-17\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}-3\\4\\-3\end{pmatrix}+\theta\begin{pmatrix}2\\3\\4\end{pmatrix},$$

as $\overrightarrow{OA}=\begin{pmatrix}2\\1\\3\end{pmatrix}$ is a point in the plane, and $\overrightarrow{AB}=\begin{pmatrix}-3\\4\\-3\end{pmatrix}$ and

$\overrightarrow{AC}=\begin{pmatrix}2\\3\\4\end{pmatrix}$ are directions parallel to the plane

Then $\begin{pmatrix}25&3&-2\\6&-4&-3\\-17&3&-4\end{pmatrix}\begin{pmatrix}\lambda\\\mu\\\theta\end{pmatrix}=\begin{pmatrix}-6\\-1\\1\end{pmatrix}$

$$\begin{vmatrix}25&3&-2\\6&-4&-3\\-17&3&-4\end{vmatrix} = 25(25)-6(-6)-17(-17) = 950$$

$$\begin{pmatrix}25&3&-2\\6&-4&-3\\-17&3&-4\end{pmatrix}^{-1} = \frac{1}{950}\begin{pmatrix}25&75&-50\\6&-134&-126\\-17&63&-118\end{pmatrix}^T$$

So $\begin{pmatrix}\lambda\\ \mu\\ \theta\end{pmatrix} = \frac{1}{950}\begin{pmatrix}25 & 6 & -17\\ 75 & -134 & 63\\ -50 & -126 & -118\end{pmatrix}\begin{pmatrix}-6\\ -1\\ 1\end{pmatrix}$

$= \frac{1}{950}\begin{pmatrix}-173\\ -253\\ 308\end{pmatrix}$,

so that $\lambda = -\frac{173}{950}$

(9) Planes

(i) Find a vector that is perpendicular to both $\begin{pmatrix}7\\ 0\\ -10\end{pmatrix}$ & $\begin{pmatrix}1\\ 3\\ -1\end{pmatrix}$

(ii) Use (i) to find the plane that passes through the points with position vectors $\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$, $\begin{pmatrix}8\\ 2\\ -7\end{pmatrix}$ & $\begin{pmatrix}0\\ -1\\ 4\end{pmatrix}$

Solution

(i) **Method 1**

$$\begin{pmatrix}7\\ 0\\ -10\end{pmatrix} \times \begin{pmatrix}1\\ 3\\ -1\end{pmatrix} = \begin{vmatrix}\underline{i} & 7 & 1\\ \underline{j} & 0 & 3\\ \underline{k} & -10 & -1\end{vmatrix} = \begin{pmatrix}30\\ -3\\ 21\end{pmatrix}$$

[From the theory of the vector product,] this is perpendicular to the given vectors (as is $\frac{1}{3}\begin{pmatrix}30\\ -3\\ 21\end{pmatrix} = \begin{pmatrix}10\\ -1\\ 7\end{pmatrix}$).

Method 2

Let the required vector be $\begin{pmatrix}1\\a\\b\end{pmatrix}$

Then $\begin{pmatrix}7\\0\\-10\end{pmatrix}.\begin{pmatrix}1\\a\\b\end{pmatrix} = 0$, so that $7 - 10b = 0$ & $b = \frac{7}{10}$

And $\begin{pmatrix}1\\3\\-1\end{pmatrix}.\begin{pmatrix}1\\a\\b\end{pmatrix} = 0$, so that $1 + 3a - b = 0$,

and $a = \frac{1}{3}\left(\frac{7}{10} - 1\right) = -\frac{1}{10}$

Thus (multiplying by 10) a suitable vector is $\begin{pmatrix}10\\-1\\7\end{pmatrix}$.

(ii) Let $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}8\\2\\-7\end{pmatrix}$ & $\begin{pmatrix}0\\-1\\4\end{pmatrix}$ represent the points A, B & C, respectively.

Then $\overrightarrow{AB} = \begin{pmatrix}7\\0\\-10\end{pmatrix}$ & $\overrightarrow{AC} = \begin{pmatrix}-1\\-3\\1\end{pmatrix} = -\begin{pmatrix}1\\3\\-1\end{pmatrix}$

From (i), a vector that is perpendicular to $\overrightarrow{AB}$ & $\overrightarrow{AC}$ (and therefore a normal to the plane) is $\begin{pmatrix}10\\-1\\7\end{pmatrix}$.

So the equation of the plane is

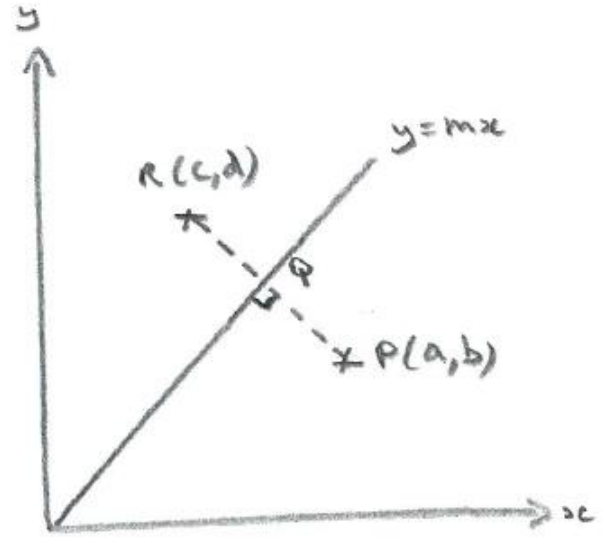
$$\underline{r}\,.\begin{pmatrix}10\\-1\\7\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}.\begin{pmatrix}10\\-1\\7\end{pmatrix} = 10 - 2 + 21 = 29$$

or (in cartesian form) $10x - y + 7z = 29$ [as $\underline{r} \equiv \begin{pmatrix}x\\y\\z\end{pmatrix}$]

(10) Points and lines

Show that the coordinates of the reflection of the point (a, b) in the line $y = mx$ are $\frac{1}{m^2+1}\begin{pmatrix} a(1-m^2)+2bm \\ 2am+b(m^2-1) \end{pmatrix}$

Solution



Referring to the diagram, let $\lambda\begin{pmatrix} 1 \\ m \end{pmatrix}$ be the point Q.

Then, as $\overrightarrow{QP}$ is perpendicular to the line $y = mx$,

$\overrightarrow{QP}.\begin{pmatrix} 1 \\ m \end{pmatrix} = 0$; ie $\begin{pmatrix} a-\lambda \\ b-\lambda m \end{pmatrix}.\begin{pmatrix} 1 \\ m \end{pmatrix} = 0$,

so that $a - \lambda + (b - \lambda m)m = 0$

$\Rightarrow \lambda(m^2+1) = a + bm$, and $\lambda = \frac{a+bm}{m^2+1}$

Then $\overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR} = \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} + \overrightarrow{PQ}$

$= \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} + \begin{pmatrix} \lambda - a \\ \lambda m - b \end{pmatrix}$

$= \begin{pmatrix} 2\lambda - a \\ 2\lambda m - b \end{pmatrix}$

$$= \frac{1}{m^2+1}\begin{pmatrix} 2(a+bm) - a(m^2+1) \\ 2m(a+bm) - b(m^2+1) \end{pmatrix}$$

$$= \frac{1}{m^2+1}\begin{pmatrix} a(1-m^2) + 2bm \\ 2am + b(m^2-1) \end{pmatrix}$$

[Note that, when $m = 1$, R is (b, a).]