Vectors Exercises - Moderate (3 pages; 21/1/21)

(1) Scalar product

Show that if $|\underline{a} - \underline{b}| = |\underline{a} + \underline{b}|$, then $\underline{a}$ & $\underline{b}$ are perpendicular.

(2) Planes

Find the cartesian form of the plane

$$\underline{r} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

(3) Lines and planes

(i)(a) Find the acute angle between the line $\frac{\boldsymbol{x}}{\boldsymbol{2}} = \frac{\boldsymbol{y+1}}{\boldsymbol{-3}} = \frac{\boldsymbol{z-2}}{\boldsymbol{1}}$ and the plane $\boldsymbol{x + y - 2z = 5}$

(b) Show that the same angle is obtained if the line is written in the form

$\frac{\boldsymbol{x}}{\boldsymbol{-2}} = \frac{\boldsymbol{y+1}}{\boldsymbol{3}} = \frac{\boldsymbol{z-2}}{\boldsymbol{-1}}$ (ie without rearranging into the form in (a))

(ii)(a) Find the acute angle between the planes $\boldsymbol{x + 4y - 3z = 7}$

and $\boldsymbol{x - y + 4z = 2}$

(b) Find the acute angle between the planes $\boldsymbol{x + 4y - 3z = 7}$ and

$\boldsymbol{-x + y - 4z = 2}$ (again, without rearranging the equation)

(4) Lines

Given that $A = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $B = \begin{pmatrix}-4\\3\\1\end{pmatrix}$, $C = \begin{pmatrix}0\\1\\2\end{pmatrix}$, $D = \begin{pmatrix}p\\4\\-4\end{pmatrix}$

(i) Write down the equations of the lines AB and CD (both extended)

(ii) Find $\overrightarrow{AB} \times \overrightarrow{CD}$

(iii) For what value of p are the lines AB and CD parallel? (2 methods)

(5) Planes

Find the plane containing the points
$(2, -1, 4)$, $(-3, 4, 2)$ and $(1, 0, 5)$, in Cartesian form

(6) Distance from point to plane

(i) Find the intersection of the line $\underline{r} = \begin{pmatrix}2\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\4\end{pmatrix}$ and the plane $3x + y + 4z = 77$

(ii) Find the shortest distance from the point $\begin{pmatrix}2\\-1\\5\end{pmatrix}$ to the plane

$3x + y + 4z = 77$

(7) Distance from point to plane

(i) Given that the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$, what is the significance of $\frac{d}{|\underline{n}|}$ if $d > 0$?

(ii) Find the equation of the plane that is parallel to $\underline{r}.\underline{n} = d$ and contains the point $\underline{p}$.

(iii) Hence deduce the formula for the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$

(8) Volume of tetrahedron

Find the volume of the tetrahedron with corners
$(2, 1, 3), (-1, 5, 0), (4, 4, 7), (8, 2, 2)$

(9) Planes

(i) Find a vector that is perpendicular to both $\begin{pmatrix} 7 \\ 0 \\ -10 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

(ii) Use (i) to find the plane that passes through the points with position vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 8 \\ 2 \\ -7 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$

(10) Points and lines

Show that the coordinates of the reflection of the point (a, b) in the line $y = mx$ are $\frac{1}{m^2+1}\begin{pmatrix} a(1-m^2)+2bm \\ 2am+b(m^2-1) \end{pmatrix}$