Vectors Exercises - Harder (Sol'ns) (21 pages; 21/1/21)

(1) Lines and planes

Find the line that is the reflection of the line $\frac{x-2}{3}=\frac{y}{4}=\frac{z+1}{1}$ in the plane $x-2y+z=4$

Solution

Let the intersection of the line and the plane be P, and suppose that Q is some other point on the line. Then we can find the reflection of Q in the plane (Q' say), by dropping a perpendicular from Q onto the plane, and then the required line will pass through P and Q'.

Writing the equation of the line as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ and substituting into the equation of the plane:

$$(2+3\lambda)-2(4\lambda)+(-1+\lambda)=4 \Rightarrow -4\lambda=3;\ \lambda=-\frac{3}{4}$$

so that P is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}-\frac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$

Setting $\lambda=1$ (say), we can take Q to be $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}+\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

Now consider the perpendicular line dropped from Q onto the plane. Its direction vector is that of the normal to the plane, and so it has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let R be the point where the perpendicular line intersects the plane. Substituting into the equation of the plane gives:

$$(5+\lambda) - 2(4-2\lambda) + (\lambda) = 4 \Rightarrow 6\lambda = 7;\ \lambda = \frac{7}{6}$$

So R is $\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \frac{7}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, and Q' will be $\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + 2\left(\frac{7}{6}\right)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{22}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix}$

Then , as P is $\begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$, the equation of the reflected line will be:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix} + \lambda\left[\begin{pmatrix} \frac{22}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix} - \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}\right] = \frac{1}{12}\begin{pmatrix} -3+\lambda(88+3) \\ -36+\lambda(-8+36) \\ -21+\lambda(28+21) \end{pmatrix}$$

$$\frac{1}{12}\begin{pmatrix} -3+91\lambda \\ -36+28\lambda \\ -21+49\lambda \end{pmatrix}$$

or, in cartesian form: $\frac{x+\frac{3}{12}}{91} = \frac{y+\frac{36}{12}}{28} = \frac{z+\frac{21}{12}}{49}$ or $\frac{x+\frac{3}{12}}{13} = \frac{y+\frac{36}{12}}{4} = \frac{z+\frac{21}{12}}{7}$

(2) Lines and planes

Find the line that is the reflection of the line $\frac{x-2}{3}=\frac{y}{4}=\frac{z+1}{1}$ in the plane $\boldsymbol{x-2y+z=4}$

Solution

Let the intersection of the line and the plane be P, and suppose that Q is some other point on the line. Then we can find the reflection of Q in the plane (Q' say), by dropping a perpendicular from Q onto the plane, and then the required line will pass through P and Q'.

Writing the equation of the line as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ and substituting into the equation of the plane:

$$(2+3\lambda)-2(4\lambda)+(-1+\lambda)=4 \Rightarrow -4\lambda=3;\ \lambda=-\frac{3}{4}$$

so that P is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}-\frac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$

Setting $\lambda=1$ (say), we can take Q to be $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}+\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

Now consider the perpendicular line dropped from Q onto the plane. Its direction vector is that of the normal to the plane, and so it has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let R be the point where the perpendicular line intersects the plane. Substituting into the equation of the plane gives:

$$\mathbf{(5+\lambda)-2(4-2\lambda)+(\lambda)=4 \Rightarrow 6\lambda=7;\ \lambda=\frac{7}{6}}$$

So R is $\begin{pmatrix}5\\4\\0\end{pmatrix}+\frac{7}{6}\begin{pmatrix}1\\-2\\1\end{pmatrix}$, and Q' will be $\begin{pmatrix}5\\4\\0\end{pmatrix}+2\left(\frac{7}{6}\right)\begin{pmatrix}1\\-2\\1\end{pmatrix}=$ $\begin{pmatrix}\frac{22}{3}\\-\frac{2}{3}\\\frac{7}{3}\end{pmatrix}$

Then , as P is $\begin{pmatrix}-\frac{1}{4}\\-3\\-\frac{7}{4}\end{pmatrix}$, the equation of the reflected line will be:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-\frac{1}{4}\\-3\\-\frac{7}{4}\end{pmatrix}+\lambda\left[\begin{pmatrix}\frac{22}{3}\\-\frac{2}{3}\\\frac{7}{3}\end{pmatrix}-\begin{pmatrix}-\frac{1}{4}\\-3\\-\frac{7}{4}\end{pmatrix}\right]=\frac{1}{12}\begin{pmatrix}-3+\lambda(88+3)\\-36+\lambda(-8+36)\\-21+\lambda(28+21)\end{pmatrix}$$

$$\frac{1}{12}\begin{pmatrix}-3+91\lambda\\-36+28\lambda\\-21+49\lambda\end{pmatrix}$$

or, in cartesian form: $\frac{x+\frac{3}{12}}{91}=\frac{y+\frac{36}{12}}{28}=\frac{z+\frac{21}{12}}{49}$ or $\frac{x+\frac{1}{4}}{13}=\frac{y+3}{4}=\frac{z+\frac{7}{4}}{7}$

(3) Lines

Find the distance between the lines $\frac{x+1}{1}=\frac{y+2}{2}$; $z=4$ and $\frac{x+3}{1}=\frac{y-6}{2}$; $z=7$, leaving your answer in exact form.

Solution

Method 1

The lines are parallel.

Choose a point on one of the lines; eg $\boldsymbol{P} = (\mathbf{-3, 6, 7})$ on the 2nd line.

To find the distance of this point from the 1st line:

A general point, Q on the 1st line is $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} \mathbf{-1} + \boldsymbol{\lambda} \\ \mathbf{-2} + \mathbf{2}\boldsymbol{\lambda} \\ \mathbf{4} \end{pmatrix}$

Then $\overrightarrow{\boldsymbol{PQ}} = \begin{pmatrix} \mathbf{-1} + \boldsymbol{\lambda} \\ \mathbf{-2} + \mathbf{2}\boldsymbol{\lambda} \\ \mathbf{4} \end{pmatrix} - \begin{pmatrix} \mathbf{-3} \\ \mathbf{6} \\ \mathbf{7} \end{pmatrix} = \begin{pmatrix} \mathbf{2} + \boldsymbol{\lambda} \\ \mathbf{-8} + \mathbf{2}\boldsymbol{\lambda} \\ \mathbf{-3} \end{pmatrix}$

We want $\overrightarrow{\boldsymbol{PQ}}$ to be perpendicular to the 1st line,

so that $\begin{pmatrix} \mathbf{2} + \boldsymbol{\lambda} \\ \mathbf{-8} + \mathbf{2}\boldsymbol{\lambda} \\ \mathbf{-3} \end{pmatrix} . \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{0} \end{pmatrix} = \mathbf{0}$

$\Rightarrow \mathbf{2} + \boldsymbol{\lambda} - \mathbf{16} + \mathbf{4}\boldsymbol{\lambda} = \mathbf{0} \quad \Rightarrow \mathbf{5}\boldsymbol{\lambda} = \mathbf{14}; \boldsymbol{\lambda} = \frac{\mathbf{14}}{\mathbf{5}}$

Then $\overrightarrow{\boldsymbol{PQ}} = \begin{pmatrix} \frac{\mathbf{24}}{\mathbf{5}} \\ -\frac{\mathbf{12}}{\mathbf{5}} \\ -\frac{\mathbf{15}}{\mathbf{5}} \end{pmatrix} = \frac{\mathbf{3}}{\mathbf{5}} \begin{pmatrix} \mathbf{8} \\ \mathbf{-4} \\ \mathbf{-5} \end{pmatrix}$ and the required distance is

$\frac{\mathbf{3}}{\mathbf{5}}\sqrt{\mathbf{64} + \mathbf{16} + \mathbf{25}}$

$= \frac{\mathbf{3}\sqrt{\mathbf{105}}}{\mathbf{5}}$

Method 2

Choose a point on each line; eg $\boldsymbol{R} = (\boldsymbol{-1}, \boldsymbol{-2}, \boldsymbol{4})$ on the 1st line, and
$\boldsymbol{P} = (\boldsymbol{-3}, \boldsymbol{6}, \boldsymbol{7})$ on the 2nd line.

Then $\overrightarrow{\boldsymbol{PR}} = \begin{pmatrix} \mathbf{2} \\ \mathbf{-8} \\ \mathbf{-3} \end{pmatrix}$ and the required distance is $\left| \frac{\begin{pmatrix} 2 \\ -8 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right|} \right|$

$$= \left| \frac{\begin{vmatrix} \underline{i} & 2 & 1 \\ \underline{j} & -8 & 2 \\ \underline{k} & -3 & 0 \end{vmatrix}}{\sqrt{5}} \right| = \frac{1}{\sqrt{5}} \left| \begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix} \right| = \frac{3}{\sqrt{5}} \left| \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right| = \frac{3}{\sqrt{5}} \sqrt{21} = \frac{3\sqrt{105}}{5}$$

(4) Lines

(i) Show the lines $\frac{x-1}{2} = \frac{y+3}{5} = \frac{z-2}{3}$ and $\frac{x}{1} = \frac{y-4}{2} = \frac{z+1}{2}$ are skew.

(ii) Find the shortest distance between the lines and identify the points on the lines at which this shortest distance occurs.

Solution

(i) The lines can be rewritten in parametric form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ -3 + 5\lambda \\ 2 + 3\lambda \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mu \\ 4 + 2\mu \\ -1 + 2\mu \end{pmatrix}$$

A point of intersection would then satisfy

$\boldsymbol{1 + 2\lambda = \mu}$ (1)
$\boldsymbol{-3 + 5\lambda = 4 + 2\mu}$ (2)
$\boldsymbol{2 + 3\lambda = -1 + 2\mu}$ (3)

Substituting from (1) into (2) & (3) gives:

$\boldsymbol{-3+5\lambda=4+2(1+2\lambda)}$ or $\boldsymbol{-9=-\lambda}$, so that $\boldsymbol{\lambda=9}$

and $\boldsymbol{2+3\lambda=-1+2(1+2\lambda)}$ or $\boldsymbol{1=\lambda}$,

and so there is no point of intersection.

Also, the direction vectors of the lines are not parallel, and so the lines are skew.

(ii) [There are various methods for finding the shortest distance, but not all of them find the points on the lines where the shortest distance occurs. The first method given below is relatively straightforward, and doesn't involve the vector product.]

Method 1

From (i), general points on the two lines are

$$\overrightarrow{\boldsymbol{OX}}=\begin{pmatrix}\boldsymbol{1+2\lambda}\\\boldsymbol{-3+5\lambda}\\\boldsymbol{2+3\lambda}\end{pmatrix}\text{ and }\overrightarrow{\boldsymbol{OY}}=\begin{pmatrix}\boldsymbol{\mu}\\\boldsymbol{4+2\mu}\\\boldsymbol{-1+2\mu}\end{pmatrix}$$

At the closest approach of the two lines, $\overrightarrow{\boldsymbol{XY}}$ will be perpendicular to both lines, so that

$$\overrightarrow{\boldsymbol{XY}}.\begin{pmatrix}\boldsymbol{2}\\\boldsymbol{5}\\\boldsymbol{3}\end{pmatrix}=\boldsymbol{0}\text{ and }\overrightarrow{\boldsymbol{XY}}.\begin{pmatrix}\boldsymbol{1}\\\boldsymbol{2}\\\boldsymbol{2}\end{pmatrix}=\boldsymbol{0}\text{, so that}$$

$$\begin{pmatrix}\boldsymbol{\mu-(1+2\lambda)}\\\boldsymbol{4+2\mu-(-3+5\lambda)}\\\boldsymbol{-1+2\mu-(2+3\lambda)}\end{pmatrix}.\begin{pmatrix}\boldsymbol{2}\\\boldsymbol{5}\\\boldsymbol{3}\end{pmatrix}=\boldsymbol{0}\text{ and}$$

$$\begin{pmatrix}\boldsymbol{\mu-(1+2\lambda)}\\\boldsymbol{4+2\mu-(-3+5\lambda)}\\\boldsymbol{-1+2\mu-(2+3\lambda)}\end{pmatrix}.\begin{pmatrix}\boldsymbol{1}\\\boldsymbol{2}\\\boldsymbol{2}\end{pmatrix}=\boldsymbol{0},$$

giving $(\boldsymbol{2\mu - 2 - 4\lambda}) + (\boldsymbol{35 + 10\mu - 25\lambda}) + (\boldsymbol{-9 + 6\mu - 9\lambda}) = \boldsymbol{0}$

or $\boldsymbol{18\mu - 38\lambda = -24}$; ie $\boldsymbol{9\mu - 19\lambda = -12}$ (1)

and $(\boldsymbol{\mu - 1 - 2\lambda}) + (\boldsymbol{14 + 4\mu - 10\lambda}) + (\boldsymbol{-6 + 4\mu - 6\lambda}) = \boldsymbol{0}$

$\boldsymbol{9\mu - 18\lambda = -7}$ (2)

Then $(\boldsymbol{1}) - (\boldsymbol{2}) \Rightarrow \boldsymbol{-\lambda = -5}$, so that $\boldsymbol{\lambda = 5}$ and, from (2),

$$\boldsymbol{\mu = \frac{1}{9}(18(5) - 7) = \frac{83}{9}}$$

So $\overrightarrow{\boldsymbol{OX}} = \begin{pmatrix} \boldsymbol{11} \\ \boldsymbol{22} \\ \boldsymbol{17} \end{pmatrix}$ and $\overrightarrow{\boldsymbol{OY}} = \frac{\boldsymbol{1}}{\boldsymbol{9}}\begin{pmatrix} \boldsymbol{83} \\ \boldsymbol{202} \\ \boldsymbol{157} \end{pmatrix}$

and $\overrightarrow{\boldsymbol{XY}} = \frac{\boldsymbol{1}}{\boldsymbol{9}}\begin{pmatrix} \boldsymbol{83 - 99} \\ \boldsymbol{202 - 198} \\ \boldsymbol{157 - 153} \end{pmatrix} = \frac{\boldsymbol{1}}{\boldsymbol{9}}\begin{pmatrix} \boldsymbol{-16} \\ \boldsymbol{4} \\ \boldsymbol{4} \end{pmatrix} = \frac{\boldsymbol{4}}{\boldsymbol{9}}\begin{pmatrix} \boldsymbol{-4} \\ \boldsymbol{1} \\ \boldsymbol{1} \end{pmatrix}$,

so that $\left|\overrightarrow{\boldsymbol{XY}}\right| = \frac{\boldsymbol{4}}{\boldsymbol{9}}\sqrt{\boldsymbol{16 + 1 + 1}} = \frac{\boldsymbol{4\sqrt{18}}}{\boldsymbol{9}} = \frac{\boldsymbol{4\sqrt{2}}}{\boldsymbol{3}}$

Method 2 (using the vector product)

If $\underline{\hat{\boldsymbol{n}}}$ is a unit vector perpendicular to both lines, then we need $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ such that $\overrightarrow{\boldsymbol{OX}} + \boldsymbol{d}\underline{\hat{\boldsymbol{n}}} = \overrightarrow{\boldsymbol{OY}}$, and the shortest distance will then be $|\boldsymbol{d}|$.

A vector perpendicular to both lines is $\begin{pmatrix} \boldsymbol{2} \\ \boldsymbol{5} \\ \boldsymbol{3} \end{pmatrix} \times \begin{pmatrix} \boldsymbol{1} \\ \boldsymbol{2} \\ \boldsymbol{2} \end{pmatrix} = \begin{vmatrix} \underline{\boldsymbol{i}} & \boldsymbol{2} & \boldsymbol{1} \\ \underline{\boldsymbol{j}} & \boldsymbol{5} & \boldsymbol{2} \\ \underline{\boldsymbol{k}} & \boldsymbol{3} & \boldsymbol{2} \end{vmatrix}$

$= \begin{pmatrix} \boldsymbol{4} \\ \boldsymbol{-1} \\ \boldsymbol{-1} \end{pmatrix}$, so that $\underline{\hat{\boldsymbol{n}}} = \frac{\boldsymbol{1}}{\sqrt{\boldsymbol{18}}}\begin{pmatrix} \boldsymbol{4} \\ \boldsymbol{-1} \\ \boldsymbol{-1} \end{pmatrix}$

Then $\overrightarrow{\boldsymbol{OX}} + \boldsymbol{d}\underline{\hat{\boldsymbol{n}}} = \overrightarrow{\boldsymbol{OY}}$ gives $\begin{pmatrix} \mathbf{1+2}\boldsymbol{\lambda} \\ \mathbf{-3+5}\boldsymbol{\lambda} \\ \mathbf{2+3}\boldsymbol{\lambda} \end{pmatrix} + \boldsymbol{D}\begin{pmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix} =$
$\begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{4+2}\boldsymbol{\mu} \\ \mathbf{-1+2}\boldsymbol{\mu} \end{pmatrix}$,
where $\boldsymbol{D} = \frac{\boldsymbol{d}}{\sqrt{\mathbf{18}}}$,

so that $\mathbf{2}\boldsymbol{\lambda} + \mathbf{4}\boldsymbol{D} - \boldsymbol{\mu} = \mathbf{-1}$ (1)
$\mathbf{5}\boldsymbol{\lambda} - \boldsymbol{D} - \mathbf{2}\boldsymbol{\mu} = \mathbf{7}$ (2)
$3\boldsymbol{\lambda} - \boldsymbol{D} - \mathbf{2}\boldsymbol{\mu} = \mathbf{-3}$ (3)

Then $(\mathbf{2}) - (\mathbf{3}) \Rightarrow \mathbf{2}\boldsymbol{\lambda} = \mathbf{10}$, so that $\boldsymbol{\lambda} = \mathbf{5}$

and (1) & (2) become $\mathbf{4}\boldsymbol{D} - \boldsymbol{\mu} = \mathbf{-11}$ (4) and $\mathbf{-}\boldsymbol{D} - \mathbf{2}\boldsymbol{\mu} = \mathbf{-18}$ (5)

Then $\mathbf{2}(\mathbf{4}) - (\mathbf{5}) \Rightarrow \mathbf{9}\boldsymbol{D} = \mathbf{-4}$, so that $|\boldsymbol{d}| = \sqrt{\mathbf{18}}|\boldsymbol{D}| = \frac{\mathbf{4}\sqrt{\mathbf{18}}}{\mathbf{9}} = \frac{\mathbf{4}\sqrt{\mathbf{2}}}{\mathbf{3}}$

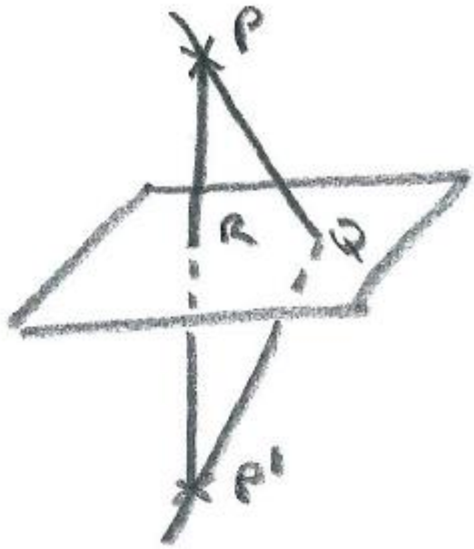
and, from (1), $\boldsymbol{\mu} = \mathbf{10} - \frac{\mathbf{16}}{\mathbf{9}} + \mathbf{1} = \frac{\mathbf{83}}{\mathbf{9}}$

and $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ can then be found, as in (i).

(5) Lines and planes

Find the reflection of the line $\frac{x-2}{3} = \frac{y+4}{1}; z = 3$ in the plane $y = 4$

Solution



Let P be $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$, say.

Q is intersection of the line and plane :

Line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

Substituting into the eq'n of the plane: $-4 + \lambda = 4 \Rightarrow \lambda = 8$

So Q is $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + 8 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 26 \\ 4 \\ 3 \end{pmatrix}$

Line PR is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

R is intersection of PR and the plane:

$-4 + \mu = 4 \Rightarrow \mu = 8$

So P' is $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + 2(8) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix}$

Eq'n of P'Q is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta[\begin{pmatrix} 26 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix}]$

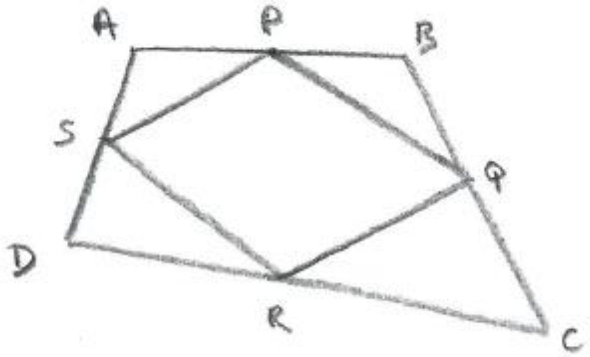
ie $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta \begin{pmatrix} 24 \\ -8 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta' \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

or $\frac{x-2}{3} = \frac{y-12}{-1}; z = 3$

(6) Problem

Use vectors to prove that the mid-points of the sides of any quadrilateral form the vertices of a parallelogram.

Solution



Referring to the diagram (where $\underline{a} = \overrightarrow{OA}\ \ etc$),

$\underline{q} - \underline{p} = \frac{1}{2}(\underline{b} + \underline{c}) - \frac{1}{2}(\underline{a} + \underline{b}) = \frac{1}{2}(\underline{c} - \underline{a})$

and $\underline{r} - \underline{s} = \frac{1}{2}(\underline{c} + \underline{d}) - \frac{1}{2}(\underline{a} + \underline{d}) = \frac{1}{2}(\underline{c} - \underline{a}) = \underline{q} - \underline{p}$

So the sides PQ & SR are of equal length and parallel.

This means that $PQRS$ is a parallelogram.

(7) Distance from point to plane

Show that the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$

Solution

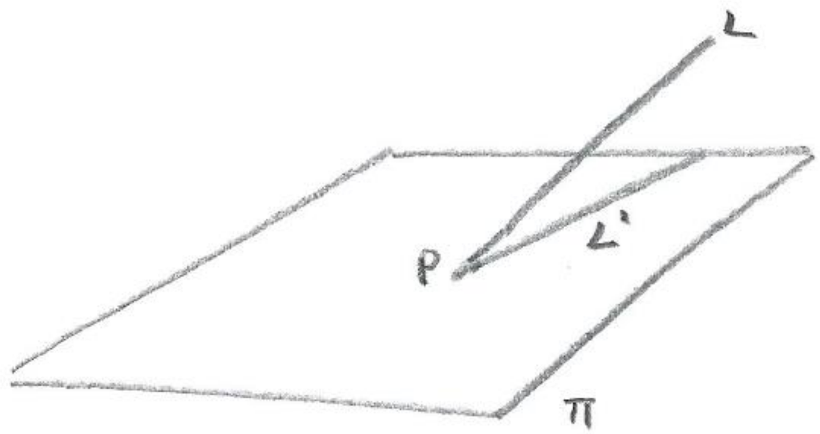
$(\underline{p} + \lambda\underline{n}).\underline{n} = d \Rightarrow \underline{p}.\underline{n} + \lambda|\underline{n}|^2 = d$

$\Rightarrow \lambda = \frac{d-\underline{p}.\underline{n}}{|\underline{n}|^2}$

So shortest distance $= |\lambda||\underline{n}| = \frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$

(8) Given the plane Π: $3x + 2y - z = 6$ and the line

$L: \underline{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, let L' be the projection of L onto Π



(i) Find the point of intersection (P) of Π & L

(ii) Find the angle between Π & L

(iii) Find a vector that is parallel to Π and perpendicular to L

(iv) Find a vector equation for L'

(v) Find the angle between L and L'

Solution

(i) $3(1+2\lambda)+2(-\lambda)-(3+\lambda)=6$

$\Rightarrow 3\lambda = 6 \Rightarrow \lambda = 2$

So P is $\begin{pmatrix} 1+4 \\ -2 \\ 3+2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$

(ii) The angle between Π & L is the angle between L and its projection onto the plane (ie the angle between L and L'), but is most easily determined by first finding the angle between L and the normal to the plane, and subtracting this from $\frac{\pi}{2}$

If θ is the angle between L and the normal to the plane, then

$$cos\theta = \frac{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}}{\sqrt{4+1+1}\sqrt{9+4+1}} = \frac{6-2-1}{\sqrt{6}\sqrt{14}} = \frac{3}{2\sqrt{21}} = \frac{3\sqrt{21}}{2(21)} = \frac{\sqrt{21}}{14}$$

The required angle is then $\arcsin\left(\frac{\sqrt{21}}{14}\right) = 19.107° = 19.1°$ (1dp)

(iii) [Note that "parallel to the plane" means parallel to a vector in the plane, and therefore perpendicular to the normal to the plane. The required vector is also perpendicular to the plane containing L and L'.]

As the required vector is perpendicular to both the normal to the plane and L, we can use the vector product:

$$\begin{pmatrix}3\\2\\-1\end{pmatrix}\times\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{vmatrix}\underline{i} & 3 & 2\\ \underline{j} & 2 & -1\\ \underline{k} & -1 & 1\end{vmatrix}=\underline{i}-5\underline{j}-7\underline{k}$$

[A useful check is that the scalar product with the original vectors is zero. Thus $3(1)+2(-5)+(-1)(-7)=0$]

(iv) L' passes through P and its direction is perpendicular to both

$\underline{i}-5\underline{j}-7\underline{k}$ (from (iii)) and the normal to the plane.

So its direction vector is

$$\begin{pmatrix}1\\-5\\-7\end{pmatrix}\times\begin{pmatrix}3\\2\\-1\end{pmatrix}=\begin{vmatrix}\underline{i} & 1 & 3\\ \underline{j} & -5 & 2\\ \underline{k} & -7 & -1\end{vmatrix}=19\underline{i}-20\underline{j}+17\underline{k}$$

So a vector equation of L' is: $\underline{r}=\begin{pmatrix}5\\-2\\5\end{pmatrix}+\lambda\begin{pmatrix}19\\-20\\17\end{pmatrix}$, from (i).

(v) If the required angle is ϕ, then

$$cos\phi=\frac{\begin{pmatrix}2\\-1\\1\end{pmatrix}.\begin{pmatrix}19\\-20\\17\end{pmatrix}}{\sqrt{4+1+1}\sqrt{361+400+289}}=\frac{38+20+17}{\sqrt{6}\sqrt{1050}}$$

$$=\frac{75}{2\sqrt{3}\sqrt{525}}=\frac{75}{2\sqrt{3}(5)\sqrt{21}}=\frac{15}{2(3)\sqrt{7}}=\frac{5\sqrt{7}}{14}$$

and hence $\phi=\arccos\left(\frac{5\sqrt{7}}{14}\right)=19.107°=19.1°$ (1dp) (as in (ii))

(9)(i) Find the intersection of the line $\underline{r} = \underline{a} + t\underline{b}$ and the plane $\underline{r}.\underline{n} = d$

(ii) Find the intersection of the line $\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and the plane $\underline{r}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -2$

Solution

(i) $(\underline{a} + t\underline{b}).\underline{n} = d \Rightarrow \underline{a}.\underline{n} + t\underline{b}.\underline{n} = d$

$\Rightarrow t = \frac{d - \underline{a}.\underline{n}}{\underline{b}.\underline{n}}$

$\Rightarrow \underline{r} = \underline{a} + (\frac{d - \underline{a}.\underline{n}}{\underline{b}.\underline{n}})\underline{b}$

(ii) Applying the result in (i):

$\underline{a}.\underline{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1$

and $\underline{b}.\underline{n} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1$

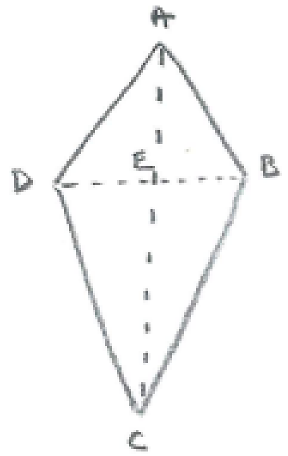
so that $\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (\frac{-2-[-1]}{-1})\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

[This can be checked by representing the line and the plane in the x-y plane.]

(10) [AEA, June 2009, Q7(d)]

In the diagram below, ABCD is a kite. Find $\overrightarrow{OD}$ if $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 4/3 \\ 7 \end{pmatrix}$,

$\overrightarrow{OB} = \begin{pmatrix} 4 \\ 4/3 \\ 2 \end{pmatrix}$ & $\overrightarrow{OC} = \begin{pmatrix} 6 \\ 16/3 \\ 2 \end{pmatrix}$



Solution

We need to take account of the special features of this case, namely that AC is perpendicular to BD (*) and bisects BD (**) (which enables D to be uniquely determined from A, B & C).

We can take an alternative route to D, in order to involve the other points, writing:

$\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BE}$ [this takes account of (**)]

To record the fact that E lies on AC, we write $\overrightarrow{BE} = \overrightarrow{BA} + \lambda\overrightarrow{AC}$

We also need to take account of (*): $\overrightarrow{BE}.\overrightarrow{AC} = 0$

Now $\overrightarrow{BA} = \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix}$, so that $\overrightarrow{BE} = \begin{pmatrix} -5+7\lambda \\ 4\lambda \\ 5-5\lambda \end{pmatrix}$

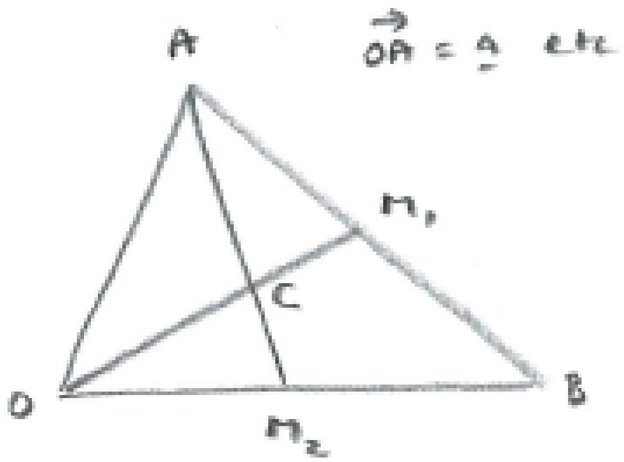
$\overrightarrow{BE}.\overrightarrow{AC} = 0$ gives $7(-5+7\lambda) + 4(4\lambda) - 5(5-5\lambda) = 0$

so that $90\lambda = 60$ and $\lambda = 2/3$

Hence $\overrightarrow{BE} = \begin{pmatrix} -1/3 \\ 8/3 \\ 5/3 \end{pmatrix}$

Then $\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BE} = \begin{pmatrix} 4 - 2/3 \\ \frac{4}{3} + 16/3 \\ 2 + 10/3 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 20/3 \\ 16/3 \end{pmatrix} = 2/3\begin{pmatrix} 5 \\ 10 \\ 18 \end{pmatrix}$

(11) Prove that the centre of mass of a triangular lamina lies 2/3 of the way along any of the medians.



Solution

Let $\overrightarrow{OC} = \lambda\overrightarrow{OM_1}$ & $\overrightarrow{AC} = \mu\overrightarrow{AM_2}$ (1)

[standard technique: represents the fact that C lies on the line OM_1]

Also, $\overrightarrow{OC} = \underline{a} + \overrightarrow{AC}$ (2) [standard technique: 2 ways of getting to the same place]

Substitute (1) into (2) ⇒ $\lambda\overrightarrow{OM_1} = \underline{a} + \mu\overrightarrow{AM_2}$ (3)

Now, $\overrightarrow{OM_1} = ½\,(\underline{a} + \underline{b})$ & $\overrightarrow{AM_2} = ½\,\underline{b} - \underline{a}$ (4)

Substitute (4) into (3) ⇒ $\frac{1}{2}\lambda(\underline{a}+\underline{b}) = \underline{a} + \mu(\frac{1}{2}\underline{b} - \underline{a})$ (5)

$\Rightarrow (\frac{\lambda}{2} + \mu - 1)\underline{a} + (\frac{\lambda}{2} - \frac{\mu}{2})\underline{b} = 0$

Provided $\underline{a}$ & $\underline{b}$ are not parallel, there is only one way of expressing a vector as a combination of $\underline{a}$ & $\underline{b}$

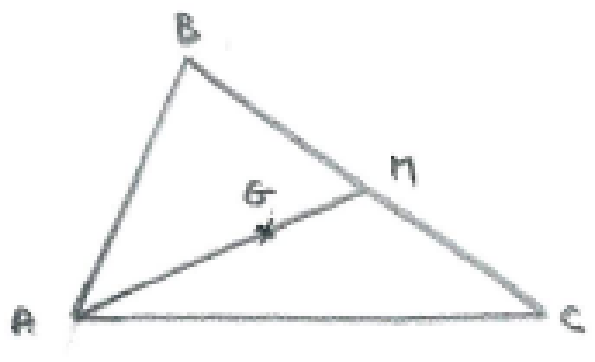
In this case, $\frac{\lambda}{2} + \mu - 1 = 0$ & $\frac{\lambda}{2} - \frac{\mu}{2} = 0$ (6)

[standard technique: equivalent to equating coefficients of $\underline{a}$ & of $\underline{b}$ in (5)]

Then (6) ⇒ $\lambda = \mu = \frac{2}{3}$

ie the centre of mass lies two-thirds of the way along any of the medians, from the relevant vertex.

(12) Given that the centre of mass of a triangular lamina lies 2/3 of the way along any of the medians, prove that it has position vector $\frac{1}{3}(\underline{a} + \underline{b} + \underline{c})$.



Solution

$\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG}$

$= \underline{a} + \frac{2}{3}\overrightarrow{AM}$

$= \underline{a} + \frac{2}{3} . ½ \, (\overrightarrow{AB} + \overrightarrow{AC})$

$= \underline{a} + \frac{1}{3} [(\underline{b} - \underline{a}) + (\underline{c} - \underline{a})]$

$= \frac{1}{3} (\underline{a} + \underline{b} + \underline{c})$

So if $\underline{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ etc, $\overrightarrow{OG} = \begin{pmatrix} \frac{1}{3}(a_x + b_x + c_x) \\ \frac{1}{3}(a_y + b_y + c_y) \end{pmatrix}$

(13) Find the angle between adjacent sloping faces of a right square-based pyramid, where the faces are equilateral triangles (as shown in Figure 1).

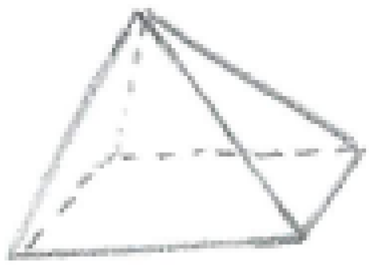


Figure 1

Solution

Without loss of generality, we can assume that the sides of the equilateral triangles forming the faces have length 2. The medians of the equilateral triangles leading to the vertex then have length $\sqrt{3}$.

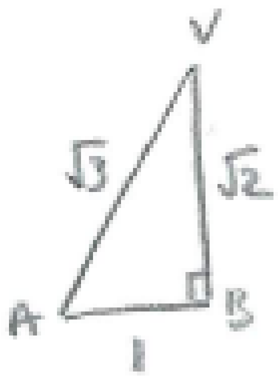


Figure 2

Referring to figure 2, we can form the right-angled triangle with corners at the vertex (V), the centre of the base of the pyramid (B), and the base of a median (A). By Pythagoras, $VB = \sqrt{2}$.

Create x and y axes along the bottom of two adjacent sloping faces of the pyramid, with z being vertical (so that the origin is at one corner of the base of the pyramid).

Then a vector equation of the plane containing one face and the y-axis is:

$$\underline{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix},$$

since the plane contains the origin, the point (0,2,0) and the point $(1,1,\sqrt{2})$, which is V (these are the 3 corners of the face).

Converting to a Cartesian equation, we have

$x = \mu$, $y = 2\lambda + \mu$ and $z = \mu\sqrt{2}$

so that $z = x\sqrt{2}$, and the equation can be written as $\sqrt{2}x + 0y - z = 0$.

Hence this face of the pyramid has direction vector $\begin{pmatrix} \sqrt{2} \\ 0 \\ -1 \end{pmatrix}$

or $\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix}$, to ensure that it is pointing away from the inside of the pyramid.

Similarly, the direction vector of the plane containing one face and the x-axis is $\begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}$.

The angle between the outward-pointing direction vectors of these faces is then given by

$$cos\theta = \frac{\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right|} = \frac{1}{3}$$

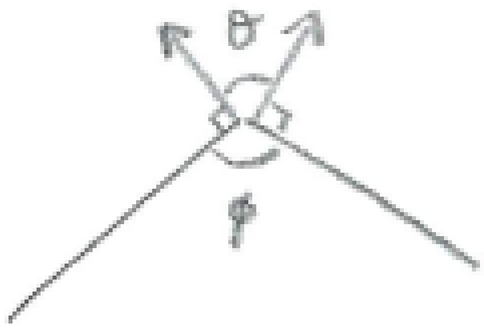


Figure 3

Referring to Figure 3, the angle that we require is $\phi = 180 - \theta$, and $cos\phi = -cos\theta = -\frac{1}{3}$

Thus $\phi = cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$ (1dp)