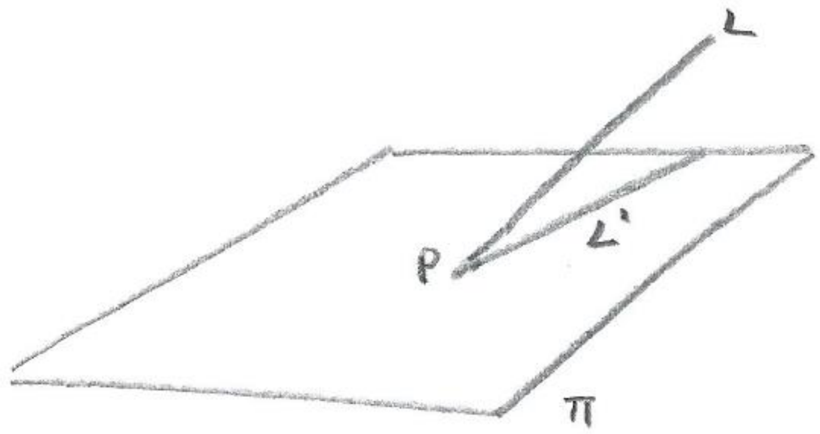
Vectors - Exercises: Part 2 (Sol'ns) (10 pages; 24/3/20)

(1***) Given the plane Π: $3x + 2y - z = 6$ and the line

$L: \underline{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, let L' be the projection of L onto Π



(i) Find the point of intersection (P) of Π & L

(ii) Find the angle between Π & L

(iii) Find a vector that is parallel to Π and perpendicular to L

(iv) Find a vector equation for L'

(v) Find the angle between L and L'

Solution

(i) $3(1 + 2\lambda) + 2(-\lambda) - (3 + \lambda) = 6$

$\Rightarrow 3\lambda = 6 \Rightarrow \lambda = 2$

So P is $\begin{pmatrix} 1+4 \\ -2 \\ 3+2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$

(ii) The angle between Π & L is the angle between L and its projection onto the plane (ie the angle between L and L'), but is most easily determined by first finding the angle between L and the normal to the plane, and subtracting this from $\frac{\pi}{2}$

If θ is the angle between L and the normal to the plane, then

$$cos\theta = \frac{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} . \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}}{\sqrt{4+1+1}\sqrt{9+4+1}} = \frac{6-2-1}{\sqrt{6}\sqrt{14}} = \frac{3}{2\sqrt{21}} = \frac{3\sqrt{21}}{2(21)} = \frac{\sqrt{21}}{14}$$

The required angle is then $\arcsin\left(\frac{\sqrt{21}}{14}\right) = 19.107° = 19.1°$ (1dp)

(iii) [Note that "parallel to the plane" means parallel to a vector in the plane, and therefore perpendicular to the normal to the plane. The required vector is also perpendicular to the plane containing L and L'.]

As the required vector is perpendicular to both the normal to the plane and L, we can use the vector product:

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{vmatrix} \underline{i} & 3 & 2 \\ \underline{j} & 2 & -1 \\ \underline{k} & -1 & 1 \end{vmatrix} = \underline{i} - 5\underline{j} - 7\underline{k}$$

[A useful check is that the scalar product with the original vectors is zero. Thus $3(1) + 2(-5) + (-1)(-7) = 0$]

(iv) L' passes through P and its direction is perpendicular to both

$\underline{i} - 5\underline{j} - 7\underline{k}$ (from (iii)) and the normal to the plane.

So its direction vector is

$$\begin{pmatrix}1\\-5\\-7\end{pmatrix}\times\begin{pmatrix}3\\2\\-1\end{pmatrix}=\begin{vmatrix}\underline{i}&1&3\\\underline{j}&-5&2\\\underline{k}&-7&-1\end{vmatrix}=19\underline{i}-20\underline{j}+17\underline{k}$$

So a vector equation of L' is: $\underline{r}=\begin{pmatrix}5\\-2\\5\end{pmatrix}+\lambda\begin{pmatrix}19\\-20\\17\end{pmatrix}$, from (i).

(v) If the required angle is ϕ, then

$$cos\phi=\frac{\begin{pmatrix}2\\-1\\1\end{pmatrix}.\begin{pmatrix}19\\-20\\17\end{pmatrix}}{\sqrt{4+1+1}\sqrt{361+400+289}}=\frac{38+20+17}{\sqrt{6}\sqrt{1050}}$$

$$=\frac{75}{2\sqrt{3}\sqrt{525}}=\frac{75}{2\sqrt{3}(5)\sqrt{21}}=\frac{15}{2(3)\sqrt{7}}=\frac{5\sqrt{7}}{14}$$

and hence $\phi=\arccos\left(\frac{5\sqrt{7}}{14}\right)=19.107°=19.1°$ (1dp) (as in (ii))

(2***)(i) Find the intersection of the line $\underline{r}=\underline{a}+t\underline{b}$ and the plane $\underline{r}.\underline{n}=d$

(ii) Find the intersection of the line $\underline{r}=\begin{pmatrix}0\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\0\end{pmatrix}$ and the plane $\underline{r}.\begin{pmatrix}1\\-1\\0\end{pmatrix}=-2$

Solution

(i) $(\underline{a}+t\underline{b}).\underline{n}=d\Rightarrow\underline{a}.\underline{n}+t\underline{b}.\underline{n}=d$

$\Rightarrow t = \frac{d-\underline{a}.\underline{n}}{\underline{b}.\underline{n}}$

$\Rightarrow \underline{r} = \underline{a} + (\frac{d-\underline{a}.\underline{n}}{\underline{b}.\underline{n}})\underline{b}$

(ii) Applying the result in (i):

$\underline{a}.\underline{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1$

and $\underline{b}.\underline{n} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1$

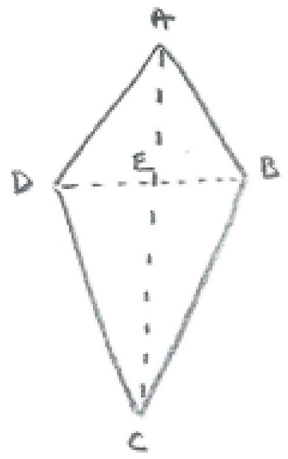
so that $\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (\frac{-2-[-1]}{-1}) \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

[This can be checked by representing the line and the plane in the x-y plane.]

(3***) [AEA, June 2009, Q7(d)]

In the diagram below, ABCD is a kite. Find $\overrightarrow{OD}$ if $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 4/3 \\ 7 \end{pmatrix}$,

$\overrightarrow{OB} = \begin{pmatrix} 4 \\ 4/3 \\ 2 \end{pmatrix}$ & $\overrightarrow{OC} = \begin{pmatrix} 6 \\ 16/3 \\ 2 \end{pmatrix}$



Solution

We need to take account of the special features of this case, namely that AC is perpendicular to BD (*) and bisects BD (**) (which enables D to be uniquely determined from A, B & C).

We can take an alternative route to D, in order to involve the other points, writing:

$\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BE}$ [this takes account of (**)]

To record the fact that E lies on AC, we write $\overrightarrow{BE} = \overrightarrow{BA} + \lambda\overrightarrow{AC}$

We also need to take account of (*): $\overrightarrow{BE}.\overrightarrow{AC} = 0$

Now $\overrightarrow{BA} = \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix}$, so that $\overrightarrow{BE} = \begin{pmatrix} -5+7\lambda \\ 4\lambda \\ 5-5\lambda \end{pmatrix}$

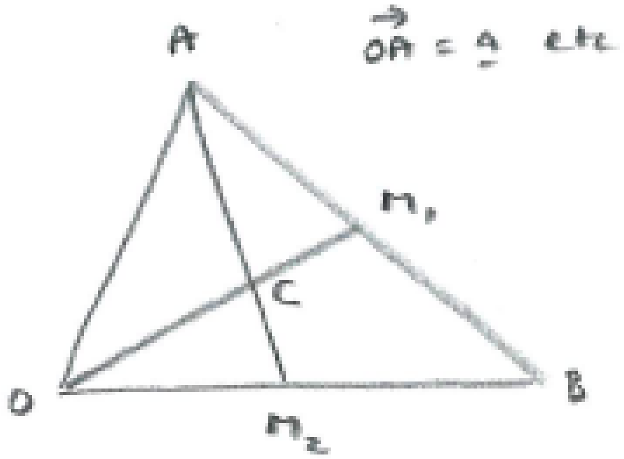
$\overrightarrow{BE}.\overrightarrow{AC} = 0$ gives $7(-5+7\lambda) + 4(4\lambda) - 5(5-5\lambda) = 0$

so that $90\lambda = 60$ and $\lambda = 2/3$

Hence $\overrightarrow{BE} = \begin{pmatrix} -1/3 \\ 8/3 \\ 5/3 \end{pmatrix}$

Then $\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BE} = \begin{pmatrix} 4 - 2/3 \\ \frac{4}{3} + 16/3 \\ 2 + 10/3 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 20/3 \\ 16/3 \end{pmatrix} = 2/3 \begin{pmatrix} 5 \\ 10 \\ 18 \end{pmatrix}$

(4***) Prove that the centre of mass of a triangular lamina lies 2/3 of the way along any of the medians.



Solution

Let $\overrightarrow{OC} = \lambda\overrightarrow{OM_1}$ & $\overrightarrow{AC} = \mu\overrightarrow{AM_2}$ (1)

[standard technique: represents the fact that C lies on the line OM_1]

Also, $\overrightarrow{OC} = \underline{a} + \overrightarrow{AC}$ (2) [standard technique: 2 ways of getting to the same place]

Substitute (1) into (2) ⇒ $\lambda\overrightarrow{OM_1} = \underline{a} + \mu\overrightarrow{AM_2}$ (3)

Now, $\overrightarrow{OM_1} = ½ (\underline{a} + \underline{b})$ & $\overrightarrow{AM_2} = ½ \underline{b} - \underline{a}$ (4)

Substitute (4) into (3) ⇒ $½ \lambda (\underline{a} + \underline{b}) = \underline{a} + \mu (½ \underline{b} - \underline{a})$ (5)

$\Rightarrow (\frac{\lambda}{2} + \mu - 1)\,\underline{a} + (\frac{\lambda}{2} - \frac{\mu}{2})\,\underline{b} = 0$

Provided $\underline{a}$ & $\underline{b}$ are not parallel, there is only one way of expressing a vector as a combination of $\underline{a}$ & $\underline{b}$

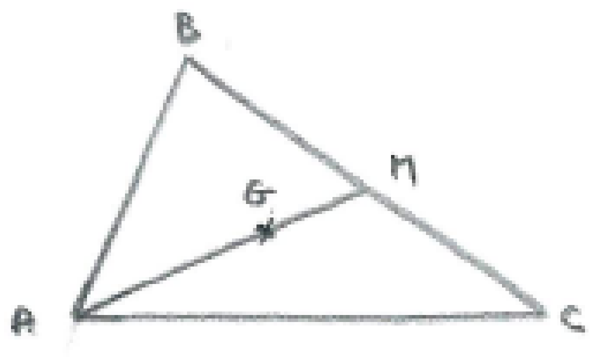
In this case, $\frac{\lambda}{2} + \mu - 1 = 0$ & $\frac{\lambda}{2} - \frac{\mu}{2} = 0$ (6)

[standard technique: equivalent to equating coefficients of $\underline{a}$ & of $\underline{b}$ in (5)]

Then (6) $\Rightarrow \lambda = \mu = \frac{2}{3}$

ie the centre of mass lies two-thirds of the way along any of the medians, from the relevant vertex.

(5***) Given that the centre of mass of a triangular lamina lies 2/3 of the way along any of the medians, prove that it has position vector $\frac{1}{3}(\underline{a} + \underline{b} + \underline{c})$.



Solution

$\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG}$

$= \underline{a} + \frac{2}{3}\overrightarrow{AM}$

$= \underline{a} + \frac{2}{3}.½(\overrightarrow{AB} + \overrightarrow{AC})$

$= \underline{a} + \frac{1}{3}[(\underline{b} - \underline{a}) + (\underline{c} - \underline{a})]$

$= \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})$

So if $\underline{a} = \binom{a_x}{a_y}$ etc, $\overrightarrow{OG} = \begin{pmatrix} \frac{1}{3}(a_x + b_x + c_x) \\ \frac{1}{3}(a_y + b_y + c_y) \end{pmatrix}$

(6***) Find the angle between adjacent sloping faces of a right square-based pyramid, where the faces are equilateral triangles (as shown in Figure 1).

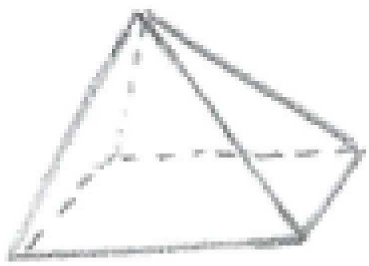


Figure 1

Solution

Without loss of generality, we can assume that the sides of the equilateral triangles forming the faces have length 2. The medians of the equilateral triangles leading to the vertex then have length $\sqrt{3}$.

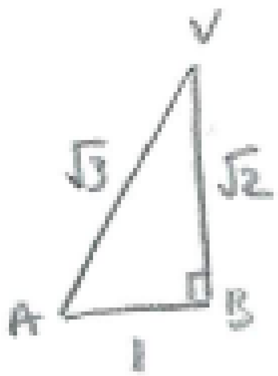


Figure 2

Referring to figure 2, we can form the right-angled triangle with corners at the vertex (V), the centre of the base of the pyramid (B), and the base of a median (A). By Pythagoras, $VB = \sqrt{2}$.

Create x and y axes along the bottom of two adjacent sloping faces of the pyramid, with z being vertical (so that the origin is at one corner of the base of the pyramid).

Then a vector equation of the plane containing one face and the y-axis is:

$$\underline{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix},$$

since the plane contains the origin, the point (0,2,0) and the point $(1,1,\sqrt{2})$, which is V (these are the 3 corners of the face).

Converting to a Cartesian equation, we have

$x = \mu$, $y = 2\lambda + \mu$ and $z = \mu\sqrt{2}$

so that $z = x\sqrt{2}$, and the equation can be written as $\sqrt{2}x + 0y - z = 0$.

Hence this face of the pyramid has direction vector $\begin{pmatrix} \sqrt{2} \\ 0 \\ -1 \end{pmatrix}$

or $\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix}$, to ensure that it is pointing away from the inside of the pyramid.

Similarly, the direction vector of the plane containing one face and the x-axis is $\begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}$.

The angle between the outward-pointing direction vectors of these faces is then given by

$$cos\theta = \frac{\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right|} = \frac{1}{3}$$

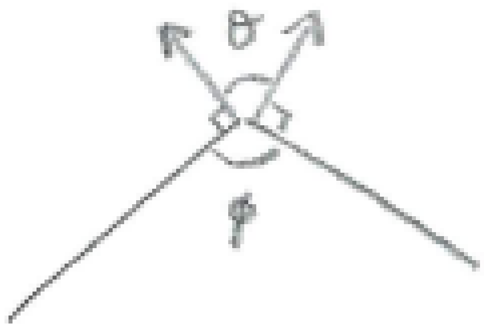


Figure 3

Referring to Figure 3, the angle that we require is $\phi = 180 - \theta$, and $cos\phi = -cos\theta = -\frac{1}{3}$

Thus $\phi = cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$ (1dp)