Vector Product - Exercises (with Sol'ns)

(13 pages; 12/5/26)

(1) Find the volume of the tetrahedron with corners
$(2, 1, 3), (-1, 5, 0), (4, 4, 7), (8, 2, 2)$

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Solution

Method 1

Label the corners as follows:

$A(2,1,3), B(-1,5,0), C(4,4,7), D(8,2,2)$

Then volume $= \frac{1}{3}.\frac{1}{2}|\overrightarrow{AB}.(\overrightarrow{AC}\times\overrightarrow{AD})|$

(based on $\frac{1}{3}$ $\times$ area of triangle ABC $\times$ perpendicular height)

and $\overrightarrow{AB}.(\overrightarrow{AC}\times\overrightarrow{AD}) = \begin{vmatrix} -3 & 2 & 6 \\ 4 & 3 & 1 \\ -3 & 4 & -1 \end{vmatrix}$

$= -3(-7) - 4(-26) - 3(-16) = 21 + 104 + 48 = 173$

So volume is $\frac{173}{6}$ units^3.

Method 2a (much longer, but good practice!)

Volume $= \frac{1}{3}$ $\times$ area of base ABC $\times$ perpendicular height

Area of base ABC $= \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

and $\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \underline{i} & -3 & 2 \\ \underline{j} & 4 & 3 \\ \underline{k} & -3 & 4 \end{vmatrix} = \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$,

so that Area of base ABC $= \frac{1}{2}\sqrt{25^2 + 6^2 + (-17)^2} = \frac{5}{2}\sqrt{38}$

The perpendicular height is the shortest distance from D to the plane ABC.

A normal to the plane ABC is $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$ (already calculated).

And the equation of the plane ABC is

$$\underline{r}.\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = 50 + 6 - 51 = 5\,,$$

taking $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ as a point in the plane.

Let the point of intersection of the perpendicular from D onto the plane ABC be P, given by the following point on the perpendicular:

$$\begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$$

As P lies in the plane ABC,

$$\left(\begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}\right).\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = 5$$

Then $178 + 950\lambda = 5$, so that $\lambda = -\frac{173}{950}$

and the perpendicular height is $|\lambda| \left| \begin{matrix} 25 \\ 6 \\ -17 \end{matrix} \right|$

$$= \frac{173}{950}\sqrt{25^2 + 6^2 + (-17)^2} = \frac{173}{950}\,.5\sqrt{38} = \frac{173}{190}\sqrt{38}$$

Hence the volume of the tetrahedron is

$$\frac{1}{3}.\frac{5}{2}\sqrt{38}\,.\frac{173}{190}\sqrt{38} = \frac{173}{6} \text{ units}^3$$

Method 2b (even longer)

As Method 2a, but determining λ as follows:

For P to be a point in the plane ABC,

$$\begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 4 \\ -3 \end{pmatrix} + \theta \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix},$$

as $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ is a point in the plane, and $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 4 \\ -3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ are directions parallel to the plane

Then $\begin{pmatrix} 25 & 3 & -2 \\ 6 & -4 & -3 \\ -17 & 3 & -4 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \theta \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \\ 1 \end{pmatrix}$

$$\begin{vmatrix} 25 & 3 & -2 \\ 6 & -4 & -3 \\ -17 & 3 & -4 \end{vmatrix} = 25(25) - 6(-6) - 17(-17) = 950$$

$$\begin{pmatrix} 25 & 3 & -2 \\ 6 & -4 & -3 \\ -17 & 3 & -4 \end{pmatrix}^{-1} = \frac{1}{950} \begin{pmatrix} 25 & 75 & -50 \\ 6 & -134 & -126 \\ -17 & 63 & -118 \end{pmatrix}^{T}$$

So $\begin{pmatrix} \lambda \\ \mu \\ \theta \end{pmatrix} = \frac{1}{950} \begin{pmatrix} 25 & 6 & -17 \\ 75 & -134 & 63 \\ -50 & -126 & -118 \end{pmatrix} \begin{pmatrix} -6 \\ -1 \\ 1 \end{pmatrix}$

$$= \frac{1}{950}\begin{pmatrix} -173 \\ -253 \\ 308 \end{pmatrix},$$

so that $\lambda = -\frac{173}{950}$

(2) Use the vector product to find the area of the triangle with corners A (1,2,3), B (4,5,6) & C (9,8,7)

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Solution

$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ & $\overrightarrow{AC} = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$

Area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\left| \begin{vmatrix} i & 3 & 8 \\ j & 3 & 6 \\ k & 3 & 4 \end{vmatrix} \right|$

$= \frac{1}{2}\ |-6i + 12j - 6k|$

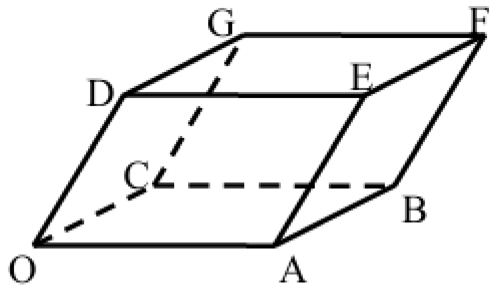
$= \frac{1}{2} \times 6 \times |i - 2j + k| = 3 \times \sqrt{1+4+1} = 3\sqrt{6}$ units^2

(3) Given that the volume of the parallelepiped shown is equal to

$|\overrightarrow{OD}.(\overrightarrow{OA} \times \overrightarrow{OC})|$, show that it also equals

(i) $|\overrightarrow{OA}.(\overrightarrow{OB} \times \overrightarrow{OD})|$

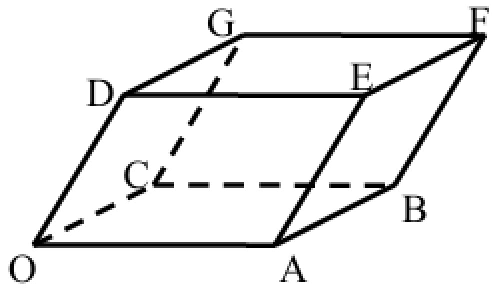
and (ii) $|\overrightarrow{OA}.(\overrightarrow{OB} \times \overrightarrow{OF})|$



(3) Given that the volume of the parallelepiped shown is equal to $|\overrightarrow{OA}.(\overrightarrow{OC} \times \overrightarrow{OD})|$, show that it also equals

(i) $\left|\overrightarrow{OA}.(\overrightarrow{OB} \times \overrightarrow{OD})\right|$

and (ii) $\left|\overrightarrow{OA}.(\overrightarrow{OB} \times \overrightarrow{OF})\right|$



Solution

(i) $\overrightarrow{OA}.(\overrightarrow{OC} \times \overrightarrow{OD}) = \overrightarrow{OA}.(\overrightarrow{AB} \times \overrightarrow{OD})$

$= \overrightarrow{OA}.([\overrightarrow{OB} - \overrightarrow{OA}] \times \overrightarrow{OD})$

$= \overrightarrow{OA}.(\overrightarrow{OB} \times \overrightarrow{OD} - \overrightarrow{OA} \times \overrightarrow{OD})$

$= \overrightarrow{OA}.(\overrightarrow{OB} \times \overrightarrow{OD}) - \overrightarrow{OA}.(\overrightarrow{OA} \times \overrightarrow{OD})$

Then, as $\overrightarrow{OA} \times \overrightarrow{OD}$ is perpendicular to $\overrightarrow{OA}$, $\overrightarrow{OA}.(\overrightarrow{OA} \times \overrightarrow{OD}) = 0$,

and so $\overrightarrow{OA}.(\overrightarrow{OC} \times \overrightarrow{OD}) = \overrightarrow{OA}.(\overrightarrow{OB} \times \overrightarrow{OD})$,

so that $\left|\overrightarrow{OA}.(\overrightarrow{OC} \times \overrightarrow{OD})\right| = \left|\overrightarrow{OA}.(\overrightarrow{OB} \times \overrightarrow{OD})\right|$, as required.

(ii) $\overrightarrow{OA}.(\overrightarrow{OB} \times \overrightarrow{OF}) = \overrightarrow{OA}.([\overrightarrow{OA} + \overrightarrow{AB}] \times [\overrightarrow{OD} + \overrightarrow{DF}])$

$= \overrightarrow{OA}.([\overrightarrow{OA} + \overrightarrow{OC}] \times [\overrightarrow{OD} + \overrightarrow{OB}])$

$= \overrightarrow{OA}.([\overrightarrow{OA} + \overrightarrow{OC}] \times [\overrightarrow{OD} + \overrightarrow{OA} + \overrightarrow{OC}])$

$= \overrightarrow{OA}.(\overrightarrow{OA} \times \overrightarrow{OD} + \overrightarrow{OA} \times \overrightarrow{OC} + \overrightarrow{OC} \times \overrightarrow{OD} + \overrightarrow{OC} \times \overrightarrow{OA})$

(as $\overrightarrow{OA} \times \overrightarrow{OA} = \underline{0}$ etc)

$= \overrightarrow{OA}.(\overrightarrow{OC} \times \overrightarrow{OD})$

(as $\overrightarrow{OA}.\left(\overrightarrow{OA} \times \overrightarrow{OD}\right) = \underline{0}$ etc, because $\overrightarrow{OA} \times \overrightarrow{OD}$ is perpendicular to $\overrightarrow{OA}$);

and so $\left|\overrightarrow{OA}.\left(\overrightarrow{OB} \times \overrightarrow{OF}\right)\right| = |\overrightarrow{OA}.\left(\overrightarrow{OC} \times \overrightarrow{OD}\right)|$, as required.

(4) (a) Show that the points

A$(24,8,-2)$, $B(9,10,2)$, $C(3,12,3)$ and D$(21,14,-4)$ are coplanar.

(b) Find the volume of the solid $OABCD$ (where O is the Origin).

(4) (a) Show that the points

A$(24,8,-2)$, $B(9,10,2)$, $C(3,12,3)$ and D$(21,14,-4)$ are coplanar.

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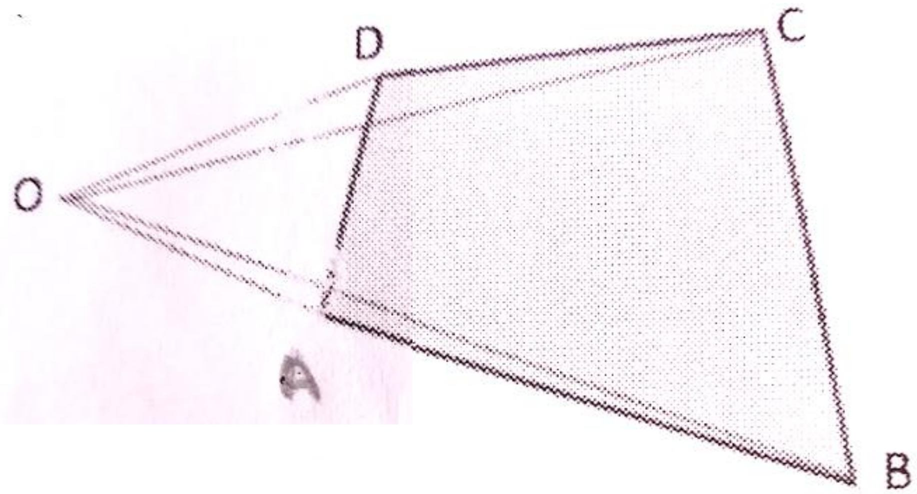
Solution

(a) **Method 1**



$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \underline{i} & -15 & -6 \\ \underline{j} & 2 & 2 \\ \underline{k} & 4 & 1 \end{vmatrix} = \begin{pmatrix} -6 \\ -9 \\ -18 \end{pmatrix}$$

and $\overrightarrow{CD} \times \overrightarrow{DA} = \begin{vmatrix} \underline{i} & 18 & 3 \\ \underline{j} & 2 & -6 \\ \underline{k} & -7 & 2 \end{vmatrix} = \begin{pmatrix} -38 \\ -57 \\ -114 \end{pmatrix}$

Now, $\begin{pmatrix} -6 \\ -9 \\ -18 \end{pmatrix} = -3\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$, whilst $\begin{pmatrix} -38 \\ -57 \\ -114 \end{pmatrix} = -19\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$,

and so $\overrightarrow{AB} \times \overrightarrow{BC}$ and $\overrightarrow{CD} \times \overrightarrow{DA}$ are parallel.

Hence the plane containing the points A, B & C has the same normal as the plane containing the points C, D & A.

Therefore these planes are parallel.

But as they have points in common (both A and C) they must therefore be the same plane; ie A, B, C & D are coplanar.

Method 2

The triangle ABC can be considered to be the base of the tetrahedron ABCD. If this tetrahedron can be shown to have a volume of zero, then D must lie in the plane of A, B & C.

The volume of the tetrahedron is $\frac{1}{3}\ |\overrightarrow{AD} . \left(\frac{1}{2}\ \overrightarrow{AB}\ \times\ \overrightarrow{AC}\right)|$

$= \frac{1}{6}\begin{vmatrix} -3 & -15 & -21 \\ 6 & 2 & 4 \\ -2 & 4 & 5 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & 5 & 7 \\ 6 & 2 & 4 \\ -2 & 4 & 5 \end{vmatrix}$

$= -\frac{1}{2}\left(-6 - 6(-3) - 2(6)\right) = 0$

So D lies in the plane of A, B & C**;** ie A, B, C & D are coplanar.

(b) Volume of $OABCD$ equals the sum of the volumes of the two tetrahedra $OABC$ and $OACD$

ie $\frac{1}{3}.\frac{1}{2}|\overrightarrow{OA}.\left(\overrightarrow{OB} \times \overrightarrow{OC}\right)| + \frac{1}{3}.\frac{1}{2}|\overrightarrow{OA}.\left(\overrightarrow{OC} \times \overrightarrow{OD}\right)|$

(based on $\frac{1}{3}$ $\times$ area of triangular base $\times$ perpendicular height)

$= \frac{1}{6}|\begin{vmatrix} 24 & 9 & 3 \\ 8 & 10 & 12 \\ -2 & 2 & 3 \end{vmatrix}| + \frac{1}{6}|\begin{vmatrix} 24 & 3 & 21 \\ 8 & 12 & 14 \\ -2 & 3 & -4 \end{vmatrix}|$

$= \frac{1}{6}|24(6) - 8(21) - 2(78)| + \frac{1}{6}|24(-90) - 8(-75) - 2(-210)|$

$= |24 - 28 - 26| + | - 360 + 100 + 70|$

$= 30 + 190 = 220\ \text{units}^3$