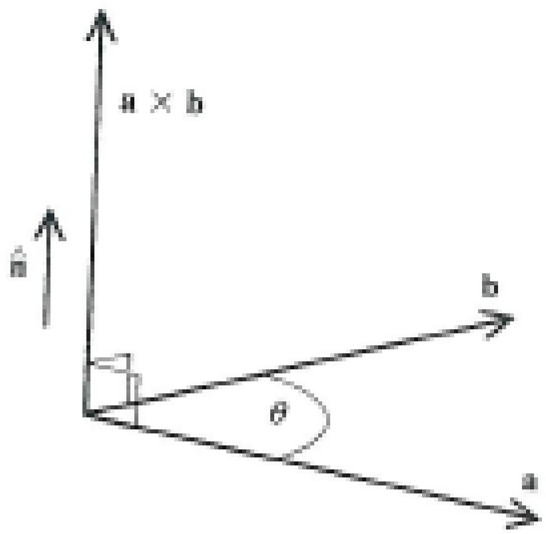
Vector Product (8 pages; 4/8/18)

(1) The vector (or 'cross') product of the (3D) vectors $\underline{a}$ and $\underline{b}$ is a vector that is perpendicular to the plane containing $\underline{a}$ and $\underline{b}$, and has magnitude $|a||b|sin\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$.

Referring to the diagram, $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}|sin\theta \, \underline{\hat{n}}$ ($\underline{\hat{n}}$ being a unit vector, with the direction shown in the diagram).



Note: One application of the vector product is in the more general treatment of moments: the moment Fd becomes $\underline{F} \times \underline{d}$

(2) The direction of $\underline{\hat{n}}$ can be obtained from the 'right-hand rule', where the curled fingers point in the direction of increasing θ ($\underline{a}$ to $\underline{b}$), and the thumb points in the direction of $\underline{\hat{n}}$.

So, if $\underline{a}$ and $\underline{b}$ are reversed, the direction of $\underline{\hat{n}}$ is reversed, and hence $\underline{b} \times \underline{a} = -\underline{a} \times \underline{b}$.

(3) It follows from the above that:

$\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = \underline{0}$

$\underline{i} \times \underline{j} = \underline{k}\ ,\ \underline{j} \times \underline{k} = \underline{i}\ ,\ \underline{k} \times \underline{i} = \underline{j}$

$\underline{j} \times \underline{i} = -\underline{k}\ ,\ \underline{k} \times \underline{j} = -\underline{i}\ ,\ \underline{i} \times \underline{k} = \underline{-j}$

(4) Whilst the scalar product provides a test for vectors being perpendicular, the vector product provides a test for their being parallel: if $\underline{a} \times \underline{b} = 0$, then $\underline{a}$ and $\underline{b}$ are parallel (assuming that neither is the zero vector); ie $\underline{b} = \lambda\underline{a}$

(5) Assuming that the distributive law applies to the vector product (which it does),

$$\left(a_1\underline{i} + a_2\underline{j} + a_3\underline{k}\right) \times \left(b_1\underline{i} + b_2\underline{j} + b_3\underline{k}\right)$$

$$= (a_2b_3 - a_3b_2)\underline{i} + (a_3b_1 - a_1b_3)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\underline{k}$$

$$= \begin{vmatrix} \underline{i} & a_1 & b_1 \\ \underline{j} & a_2 & b_2 \\ \underline{k} & a_3 & b_3 \end{vmatrix} \text{ or } \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note: The $\underline{i}$ component $(a_2b_3 - a_3b_2)$ involves only 2s & 3s; the 1st term (23) is in the 'forwards' direction; the 2nd (32) is in the 'backwards' direction.

Example

$$\left(4\underline{i} + 3\underline{j} + 2\underline{k}\right) \times \left(2\underline{i} - \underline{j} + 5\underline{k}\right)$$

$$= \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

$$= \begin{vmatrix} \underline{i} & 4 & 2 \\ \underline{j} & 3 & -1 \\ \underline{k} & 2 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} \underline{i} - \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} \underline{j} + \begin{vmatrix} 4 & 2 \\ 3 & -1 \end{vmatrix} \underline{k}$$

$$= 17\underline{i} - 16\underline{j} - 10\underline{k}$$

Check:

We expect $17\underline{i} - 16\underline{j} - 10\underline{k}$ to be perpendicular to both of the original vectors.

$$\begin{pmatrix} 17 \\ -16 \\ -10 \end{pmatrix} . \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 68 - 48 - 20 = 0$$

$$\begin{pmatrix} 17 \\ -16 \\ -10 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 34 + 16 - 50 = 0$$

Example: Find a unit vector perpendicular to the vectors $4\underline{i} + 3\underline{j} + 2\underline{k}$ and $2\underline{i} - \underline{j} + 5\underline{k}$

Solution

$$\left|17\underline{i} - 16\underline{j} - 10\underline{k}\right| = \sqrt{17^2 + (-16)^2 + (-10)^2} = \sqrt{645},$$

so that the required unit vector is

$\frac{1}{\sqrt{645}}(17\underline{i} - 16\underline{j} - 10\underline{k})$

(6) The vector product can't properly be used to find the angle between two vectors.

Example: Find the angle between $\underline{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

Method A: scalar product

$\underline{a}.\underline{b} = |\underline{a}|\ |\underline{b}|cos\theta$

Also, $\underline{a}.\underline{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = 1 + 2 + 4 = 7$

So $cos\theta = \frac{7}{\sqrt{3}\sqrt{21}} = \frac{\sqrt{7}}{3} \Rightarrow \quad \theta = 0.49088$ (5sf)

Method B: vector product

$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}||sin\theta|$

[To make any progress, we have to consider the magnitude of the vector product. But this leads to spurious solutions.]

Also, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{vmatrix} \underline{i} & 1 & 1 \\ \underline{j} & 1 & 2 \\ \underline{k} & 1 & 4 \end{vmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

so that $|\underline{a} \times \underline{b}| = \sqrt{4 + 9 + 1} = \sqrt{14}$

and $|sin\theta| = \frac{\sqrt{14}}{\sqrt{3}\sqrt{21}} = \frac{\sqrt{2}}{3}$

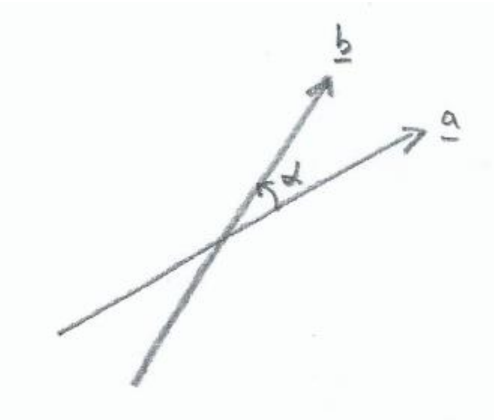
$\Rightarrow sin\theta = \frac{\sqrt{2}}{3}$, so that $\theta = 0.49088$ or $\pi - 0.49088$

or $sin\theta = -\frac{\sqrt{2}}{3}$, so that $\theta = -0.49088$ or $\pi - (-0.49088)$

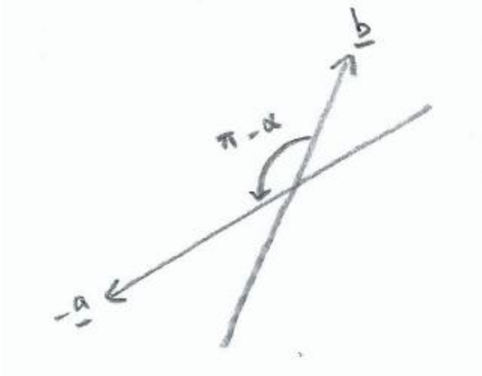
We know, from using the scalar product, that $\theta = 0.49088$ is the acute angle between $\underline{a}$ and $\underline{b}$. $\theta = -0.49088$ arises because $|\underline{b} \times \underline{a}| = |\underline{a} \times \underline{b}|$. With $\underline{b} \times \underline{a}$ we are just measuring the angle in the opposite direction, so once again the required angle is 0.49088.

$\theta = \pi - 0.49088$ arises from $\underline{b} \times (-\underline{a})$, and so there is an ambiguity (from the information we have, $\pi - 0.49088$ could be the required angle) - see the diagrams below, where $\alpha = 0.49088$

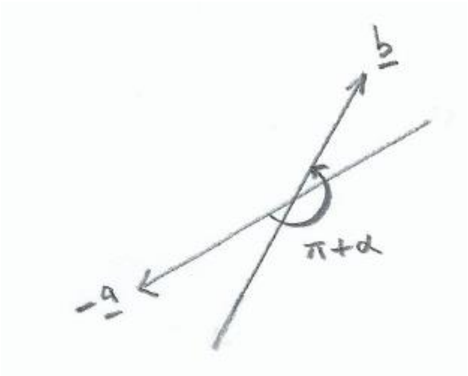
$\theta = \pi - (-0.49088) = \pi + 0.49088$ arises from $(-\underline{a}) \times \underline{b}$, and the required angle would be $2\pi - (\pi + 0.49088) = \pi - 0.49088$ again.



α arises from $\underline{a} \times \underline{b}$



$\pi - \alpha$ arises from $\underline{b} \times (-\underline{a})$



$\pi + \alpha$ arises from $(-\underline{a}) \times \underline{b}$

[There are other possibilities; eg $(-\underline{a}) \times (-\underline{b})$ and $(-\underline{b}) \times \underline{a}$, but they produce the same angles.]

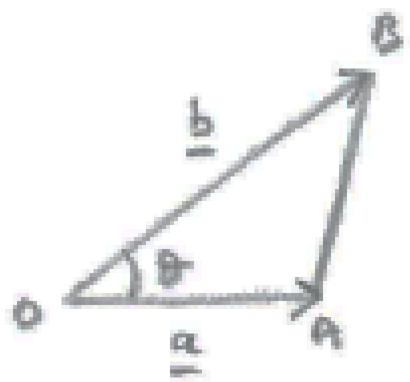
In conclusion, if the vector product reveals that $|sin\theta| = k$, then the angle between the vectors could be either $sin^{-1}k$ or $\pi - sin^{-1}k$. If only the acute angle between the two vectors is required, then the answer is $sin^{-1}k$.

(7) Areas

(i) Triangle

$= \frac{1}{2}\ |\underline{a}||\underline{b}|sin\theta$

$= \frac{1}{2}\ |\underline{a} \times \underline{b}|$



Example: Find the area of the triangle with corners A (1,2,3), B (4,5,6) & C (9,8,7)

Solution

$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ & $\overrightarrow{AC} = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$

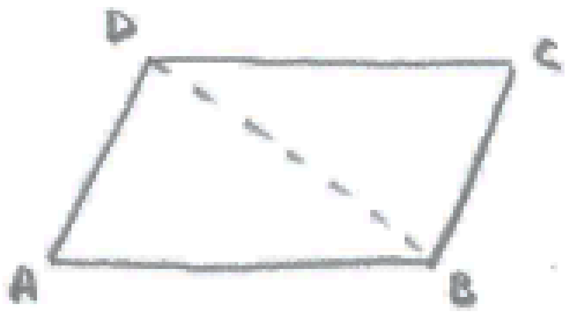
Area = $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\left|\begin{vmatrix} \underline{i} & 3 & 8 \\ \underline{j} & 3 & 6 \\ \underline{k} & 3 & 4 \end{vmatrix}\right|$

$= \frac{1}{2}\ \left|-6\underline{i} + 12\underline{j} - 6\underline{k}\right|$

$= \frac{1}{2} \times 6 \times \left|\underline{i} - 2\underline{j} + \underline{k}\right| = 3 \times \sqrt{1 + 4 + 1} = 3\sqrt{6}$

(ii) Parallelogram

$= \left|(\underline{b} - \underline{a}) \times (\underline{d} - \underline{a})\right|$



[Area is twice that of the triangle ABD]

$= |\underline{b} \times \underline{d} - \underline{b} \times \underline{a} - \underline{a} \times \underline{d} + \underline{a} \times \underline{a}|$

$= |\underline{a} \times \underline{b} + \underline{b} \times \underline{d} + \underline{d} \times \underline{a}|$

[Note that ABD is anti-clockwise.]

(8) Proof that the vector product is distributive over vector addition

ie that $\underline{a} \times (\underline{b} + \underline{c}) = (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c})$

or that $\underline{a} \times (\underline{b} + \underline{c}) - (\underline{a} \times \underline{b}) - (\underline{a} \times \underline{c}) = 0$

We will show that $\underline{\mathrm{r}}.\left[\underline{a} \times (\underline{b} + \underline{c}) - (\underline{a} \times \underline{b}) - (\underline{a} \times \underline{c})\right] = 0$ for any $\underline{r}$ (which implies the required result)

LHS $= \underline{r}.\left[\underline{a} \times (\underline{b} + \underline{c})\right] - \underline{\mathrm{r}}.(\underline{a} \times \underline{b}) - \underline{\mathrm{r}}.(\underline{a} \times \underline{c})$

by distributivity of the scalar product over vector addition

$= (\underline{b} + \underline{c}).(\underline{r} \times \underline{a}) - \underline{b}.(\underline{\mathrm{r}} \times \underline{a}) - \underline{c}.(\underline{\mathrm{r}} \times \underline{a})$, by cyclic interchange

$= \underline{b}.(\underline{r} \times \underline{a}) + \underline{c}.(\underline{r} \times \underline{a}) - \underline{b}.(\underline{\mathrm{r}} \times \underline{a}) - \underline{c}.(\underline{\mathrm{r}} \times \underline{a})$

by distributivity of scalar product

$= 0$

(9) To find a vector perpendicular to a given 3D vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$: just take the vector product with eg $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, to give $\begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}$ [as $\underline{a} \times \underline{b}$ will be perpendicular to $\underline{a}$]

[As can be seen, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}.\begin{pmatrix} 0 \\ c \\ -b \end{pmatrix} = 0$]