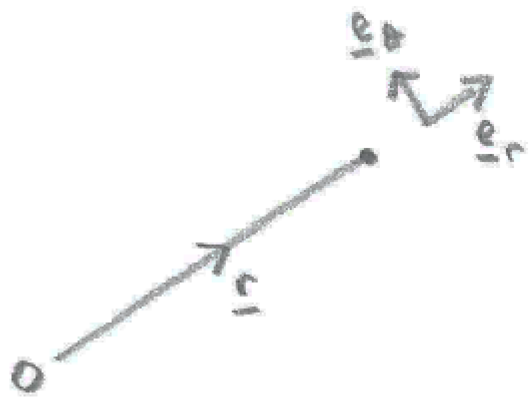
Circular Motion - Variable Speed (13 pages; 30/9/15)

(mainly relating to vertical circular motion)

(1) Velocity & acceleration vectors



$\underline{r} = rcos\theta\,\underline{i} + rsin\theta\,\underline{j} = r\underline{e_r}$ (referring to the diagram)

As r is assumed to be constant:

$\underline{v} = \frac{d\underline{r}}{dt} = -rsin\theta(\dot{\theta})\underline{i} + rcos\theta(\dot{\theta})\underline{j} = (r\dot{\theta})\left(-sin\theta\underline{i} + cos\theta\underline{j}\right)$

$= r\dot{\theta}\underline{e_\theta}$ (**Exercise**: confirm the last step)

$\underline{a} = \frac{d\underline{v}}{dt} = (r\ddot{\theta})(-sin\theta\underline{i} + cos\theta\underline{j}) + (r\dot{\theta})(-cos\theta(\dot{\theta})\underline{i} - sin\theta(\dot{\theta})\underline{j})$

$= r\ddot{\theta}\underline{e_\theta} - r(\dot{\theta})^2\underline{e_r}$

Components of acceleration:

radial $-r(\dot{\theta}^2)$ or $-r\left(\frac{v}{r}\right)^2 = -\frac{v^2}{r}$ tangential $r\ddot{\theta}$ or $\frac{dv}{dt}$

($s = r\theta$, $v = r\dot{\theta}$, $\frac{dv}{dt} = r\ddot{\theta}$, as r is constant)

magnitude of acceleration: $\sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$

(2) Exercise

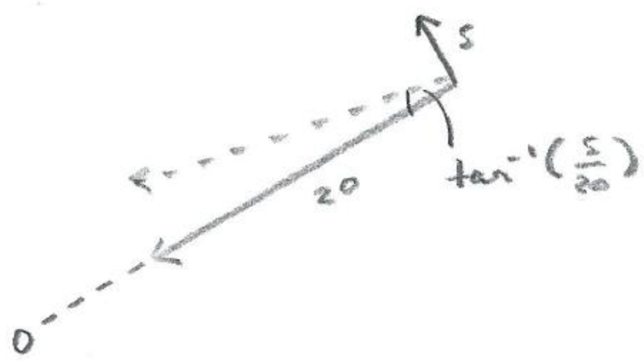
Find the magnitude and direction of the resultant force needed for circular motion when mass = 2g, $r = 5m$, $v = 10ms^{-1}$ and $\frac{dv}{dt} = 5ms^{-2}$

Solution

$|\underline{a}| = \sqrt{\left(\frac{100}{5}\right)^2 + 5^2} = \sqrt{425} = 20.6\, ms^{-2}$

Hence magnitude of force is $0.002 \times 20.6 = 0.0412\ N$

Direction is $tan^{-1}\left(\frac{5}{20}\right) = tan^{-1}\left(\frac{1}{4}\right) = 14.0°$ to the radius



(3) Motion in a vertical circle

Types of situation:

(a) particle on end of string

(b) particle on end of rod

(c) particle threaded on a wire

(d) particle inside hoop

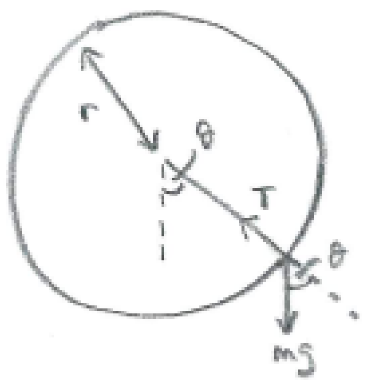
(e) particle outside hoop

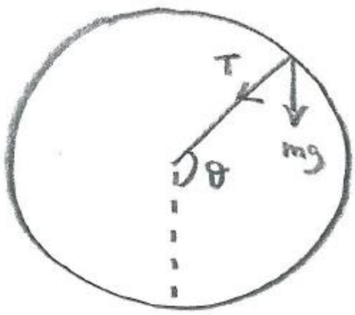
We are not usually interested in tangential acceleration for vertical circle questions.

(4): (a) particle on end of string

Tension, towards centre

(For the following diagrams, the particle itself isn't visible.)





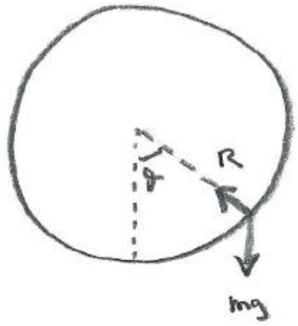
Possibilities:

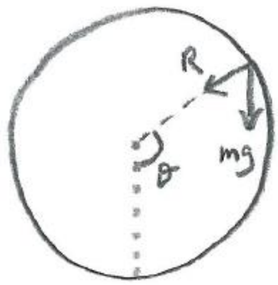
- particle could come to a halt before reaching horizontal position (T will still be >0)

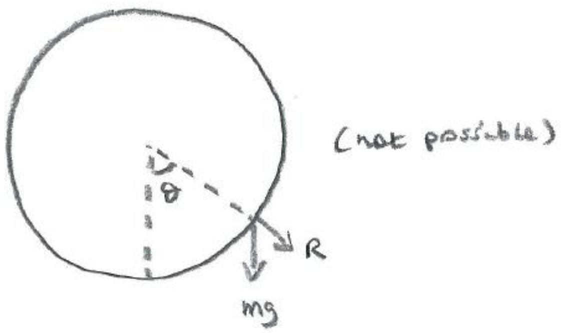
- string could become slack before reaching top (after reaching horizontal), and circular motion ceases (particle then behaves as a projectile)

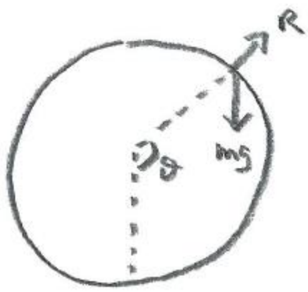
(5): (b) particle threaded on wire

Normal reaction either towards the centre or away from it









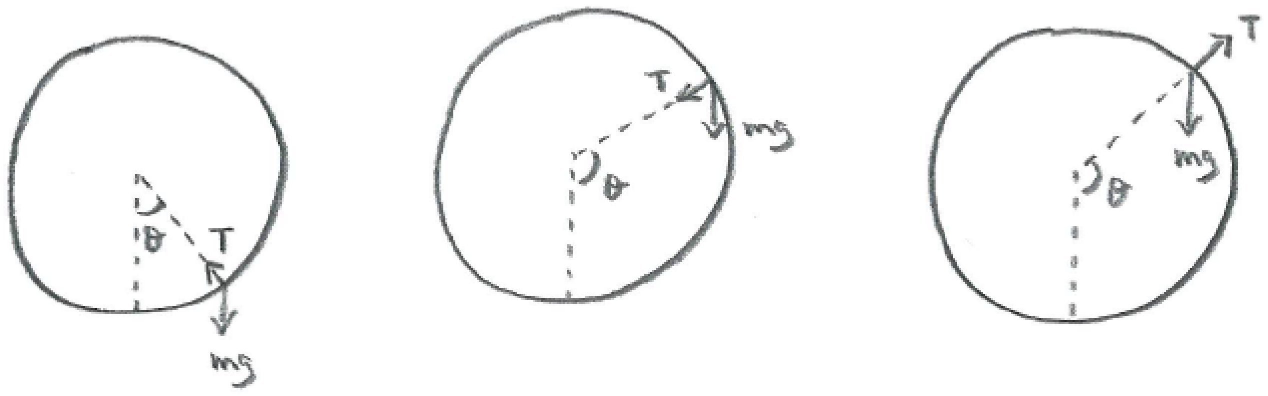
Of these situations, the bottom left one is not possible, as there can be no net force towards the centre.

Possibilities:

- particle could come to a halt before reaching top

(6): (c) particle on end of rod

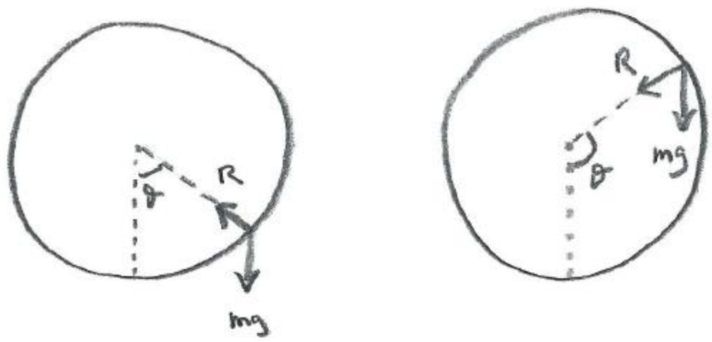
Equivalent to (b) particle threaded on wire



Note that, for the 3rd case, the rod is under compression (the force T away from the centre is being applied to the particle by the rod; so, by Newton's 3rd law, there is a force of T towards the centre being applied to the rod by the particle).

(7): (d) particle inside hoop

Normal reaction, towards centre

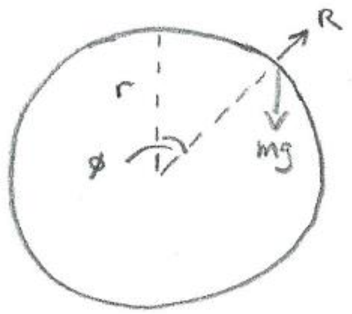


Equivalent to (a) particle on end of string

(8): (e) particle outside hoop

Normal reaction, away from centre

(usual to measure angle from the top)



ϕ can't be greater than 90° (otherwise there can be no net force towards the centre)

Possibilities:

- particle will leave surface at some point

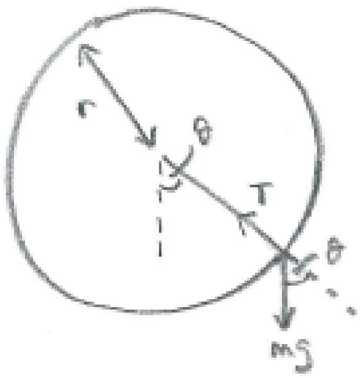
(9) Summary of Situations

(A) particle on end of string (a) / inside hoop (d)

(B) particle threaded on wire (b) / on end of rod (c)

(C) particle outside hoop (e)

(10) Equation 1 - for situations A or B

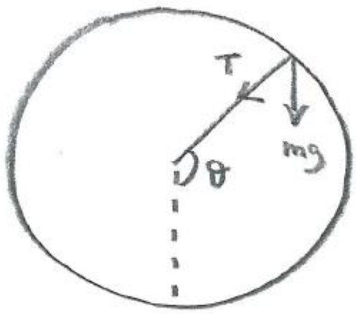


$\theta \leq \frac{\pi}{2}$: $T - mgcos\theta = \frac{mv^2}{r}$

$\theta \geq \frac{\pi}{2}$: $T + mgcos(\pi - \theta) = \frac{mv^2}{r}$

and $\cos(\pi - \theta) = -cos\theta$

So $T - mgcos\theta = \frac{mv^2}{r}$ again



(11) Exercise (same situations)

When is T maximised?

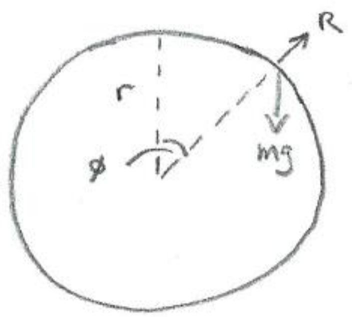
Solution

$T - mgcos\theta = \frac{mv^2}{r} \Rightarrow T = \frac{mv^2}{r} + mgcos\theta$

So T is maximised when $\theta = 0°$

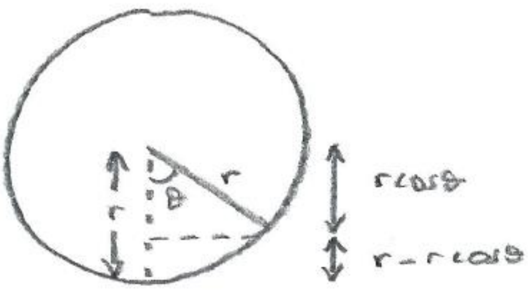
(12) Equation 1 - for situation C (particle outside hoop)

$\text{mgcos}\phi - R = \frac{mv^2}{r}$



(13) Equation 2: Conservation of Energy

Loss of KE = Gain in PE (or vice versa)



Example - Situation B (particle threaded on wire / on end of rod)

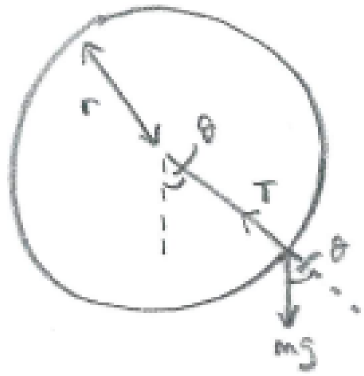
(Note that, for this situation, we don't need to worry about the reaction/tension becoming zero.)

Referring to the diagram below, if a particle of mass m is at rest at the top of the circle, and is then nudged to one side, find its speed when $\theta = 90°$. Also find the tension T in this position.

By Conservation of Energy, $PE_0 + KE_0 = PE_1 + KE_1$

Taking the zero of PE to be when $\theta = 0°$:

$mg(2r) + 0 = mgr + \frac{1}{2}mv^2 \Rightarrow v^2 = 2gr \Rightarrow v = \sqrt{2gr}$



$T - mgcos\theta = \frac{mv^2}{r}$ & $v = \sqrt{2gr}$

$\Rightarrow T = mgcos90° + \frac{m(2gr)}{r} = 2mg$

(14) Exercise (same example)

Find the speed and tension when $\theta = 0°$

Solution

By Conservation of Energy, $mg(2r) + 0 = 0 + \frac{1}{2}mv^2$

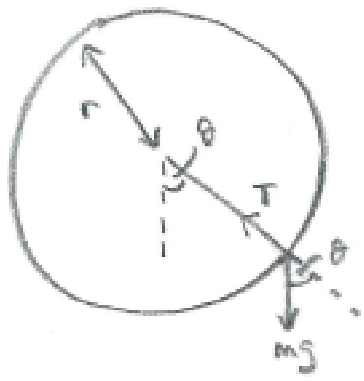
$\Rightarrow v^2 = 4gr \Rightarrow v = 2\sqrt{gr}$

Then $T - mgcos\theta = \frac{mv^2}{r}$ & $v = 2\sqrt{gr}$

$\Rightarrow T = mgcos0° + \frac{m(4gr)}{r} = 5mg$

(15) Example: Situation B (particle threaded on wire / on end of rod)

Let the particle start at the bottom with speed u.



To find the minimum value of u such that the top of the circle is reached:

We need $v > 0$ throughout.

Conservation of energy:

$\frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 + mgr(1 - cos\theta)$

$v > 0 \Rightarrow v^2 = u^2 - 2gr(1 - cos\theta) \; > \; 0$

$\Rightarrow u^2 > 2gr(1 - cos\theta)$

The greatest value of the RHS is $4gr$ ($when\ \theta = 180°$).

So we require $u^2 \geq 4gr\ ; ie\ u \geq 2\sqrt{gr}$

(Note that this agrees with (13).)

(16) Exercise (same example)

Find the speed needed at the bottom, in order for the particle to reach (a) $\theta = 90°$ $(b)\ \theta = 120°$ before dropping back.

Solution

(a) As before, $v > 0 \Rightarrow u^2 > 2gr(1 - cos\theta)$

For $0 \leq \theta \leq 90°$, the greatest value of RHS $= 2gr(1 - 0)$ (when $\theta = 90°$).

So we require $u^2 \geq 2gr$; ie $u \geq \sqrt{2gr}$

(b) For $0° \leq \theta \leq 120°$, the greatest value of RHS $= 2gr(1 - (-0.5))$ (when $\theta = 120°$).

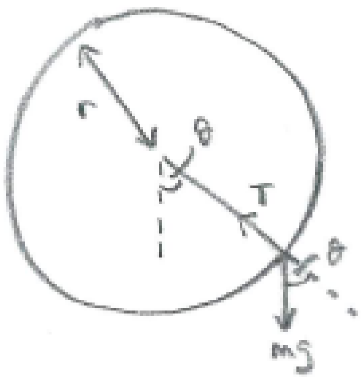
So we require $u^2 \geq 3gr$; ie $u \geq \sqrt{3gr}$

(17) Example: Situation A (particle on end of string / inside hoop)

From (10),

$T - mgcos\theta = \frac{mv^2}{r}$ $\quad (T\ has\ to\ be \geq 0)$

$\Rightarrow T = \frac{mv^2}{r} + mgcos\theta$



For $0 \leq \theta < 90°$, $cos\theta > 0$,

so that $T > \frac{mv^2}{r}$

and hence $v = 0$ occurs before $T = 0$

For $90° < \theta \leq 180°$, $cos\theta < 0$,

so that $T < \frac{mv^2}{r}$

and hence T $= 0$ occurs before $v = 0$

(at $\theta = 90°$, $v = 0 \Leftrightarrow T = 0$)

So to reach $\boldsymbol{\theta < 90°}$, we require $\boldsymbol{v > 0}$ (as for situation B)

whilst to reach $\boldsymbol{\theta > 90°}$, we require T>0

As before, Conservation of Energy $\Rightarrow$

$\frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 + mgr(1 - cos\theta)$

$\Rightarrow v^2 = u^2 - 2gr(1 - cos\theta)$

and $T - mgcos\theta = \frac{mv^2}{r}$

$T > 0 \Rightarrow \frac{mv^2}{r} + \text{mgcos}\theta > 0$

$\Rightarrow \frac{m}{r}(u^2 - 2gr(1 - cos\theta) + \text{grcos}\theta) > 0$

$\Rightarrow u^2 - 2gr + 3\text{grcos}\theta > 0$

$\Rightarrow u^2 > gr(2 - 3cos\theta)$

(18) Exercise (same situation)

What value must u have to ensure that the top of the circle is reached?

Solution

$u^2 > gr(2 - 3cos\theta)$

Greatest value of RHS is $gr\big(2 - 3(-1)\big) = 5gr$ (when $\theta = 180°$),

so required $u = \sqrt{5gr}$

(19) Breakdown of circular motion: Situation A (particle on end of string / inside hoop)

This is where the string goes slack ($T = 0$) or $R = 0$, in the case of the particle inside a hoop (Situation C, where the particle is outside a hoop, is dealt with in the next section).

For the case of the string, we start with the usual equations:

$T - mgcos\theta = \frac{mv^2}{r}$ and $\frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 + mgr(1 - cos\theta)$

Then setting $T = 0$:

$mv^2 = -mgrcos\theta$ and $mv^2 = mu^2 - 2mgr(1 - cos\theta)$

$\Rightarrow -mgrcos\theta = mu^2 - 2mgr(1 - cos\theta)$

$\Rightarrow u^2 = 2gr - 3grcos\theta$

$\Rightarrow 3grcos\theta = 2gr - u^2$

$\Rightarrow cos\theta = \frac{2gr - u^2}{3gr}$

eg if $u^2 = 3gr,\ cos\theta = -\frac{1}{3} \Rightarrow \theta = 109.5°$

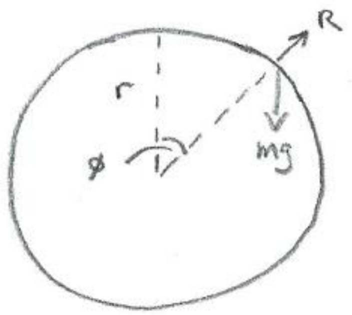
ie when the speed at the bottom, $u = \sqrt{3gr}$, the particle reaches $\theta = 109.5°$ (measured from the bottom).

Note: The particle behaves as a projectile once the string has gone slack.

The same equations apply where the particle is inside a hoop, with R replacing T.

(20) Breakdown of circular motion: Situation C (particle outside hoop)

Exercise: If the particle has speed u at the top, find the angle at which it leaves the circle, in terms of u, g & r , referring to the diagram.



Solution

$mgcos\phi - R = \frac{mv^2}{r}$

By Conservation of Energy (taking the top as PE=0),

$\frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 - mgr(1 - cos\phi)$

$\Rightarrow u^2 = v^2 - 2gr(1 - cos\phi)$

$Then\ \ R = 0 \Rightarrow gcos\phi = \frac{v^2}{r} = \frac{1}{r}(u^2 + 2gr(1 - cos\phi))$

$\Rightarrow grcos\phi = u^2 + 2gr(1 - cos\phi)$

$\Rightarrow 3grcos\phi = u^2 + 2gr$

$\Rightarrow cos\phi = \frac{u^2+2gr}{3gr}$ and $\phi = cos^{-1}(\frac{u^2+2gr}{3gr})$

(21) Exercise (same example)

Find (a) the greatest possible value of ϕ

(b) the value of u if the particle leaves the circle straightaway (ie at the top)

Solution

(a) ϕ is maximised when $cos\phi$ is minimised

$cos\phi > \frac{2gr}{3gr} = \frac{2}{3} \Rightarrow \phi = 48.2°$

ie $\phi < 48.2°$ (3sf)

(b) $\phi = 0 \Rightarrow \frac{u^2+2gr}{3gr}$=1 $\Rightarrow u^2 = gr$; ie $u = \sqrt{gr}$