Useful Results - Mechanics (4 pages; 9/8/23)

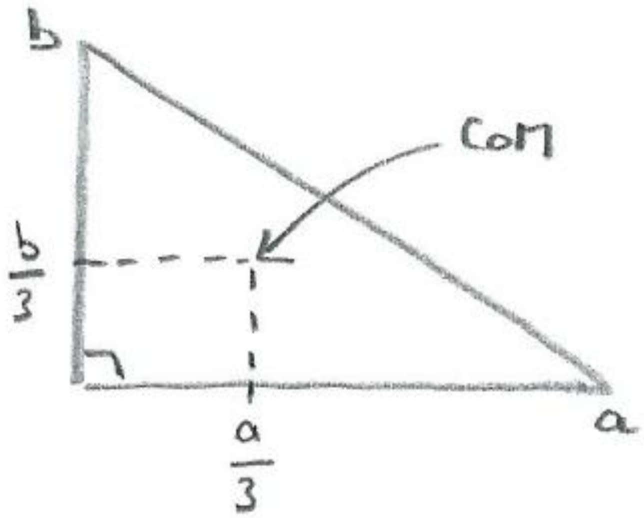
(1) Centre of Mass

Triangular lamina

(a) average of coordinates of vertices

(b) $\frac{2}{3}$ along median from vertex

(c) right-angled triangular lamina: see diagram below



Solid cone or pyramid of height h

$\frac{h}{4}$ from base (on line of symmetry)

Hollow cone or pyramid of height h

$\frac{h}{3}$ from base (on line of symmetry)

Sector of circle (radius r, angle 2θ at centre)

$\frac{2rsin\theta}{3\theta}$ [As $\theta \to 0$, $\frac{2rsin\theta}{3\theta} \to \frac{2r}{3}$ (as $\frac{sin\theta}{\theta} \to 1$) ; as $\theta \to \frac{\pi}{2}$, CoM moves nearer the centre, and $\frac{sin\theta}{\theta}$ reduces]

Arc of circle (radius r, angle 2θ at centre)

$\frac{rsin\theta}{\theta}$ [As $\theta \to 0, \frac{rsin\theta}{\theta} \to r$; as $\theta \to \frac{\pi}{2}$, CoM moves nearer the centre]

(2) Projectiles

Cartesian equation: $y = xtan\theta - \frac{gx^2}{2u^2cos^2\theta}$

Maximum height: $\frac{u^2sin^2\theta}{2g}$

Time to reach maximum height: $\frac{usin\theta}{g}$

Range: $\frac{sin2\theta.u^2}{g}$

(3) SHM

$\ddot{x} = -\omega^2 x$

$x = asin(\omega t + \epsilon)$

$v^2 = \omega^2(a^2 - x^2)$

(4) Rigid bodies

Moment of inertia, $I = \sum m_i {r_i}^2$

KE $= \frac{1}{2}I\omega^2$ (where ω or $\dot{\theta}$ is the angular velocity)

Angular momentum $= I\omega$

Total moments of forces , $C = I\ddot{\theta}$

Work done $= \int C\, d\theta$

(5) Correspondence between angular and linear quantities

Linear	Angular	Notes
s	θ	
v	$\dot{\theta}$	
a	$\ddot{\theta}$	
m	I	I is the moment of inertia (see separate note)
mv	$I\dot{\theta}$	$I\dot{\theta}$ is angular momentum (often denoted by L)
$\frac{1}{2}mv^2$	$\frac{1}{2}I\left(\dot{\theta}\right)^2$	kinetic energy
$F = ma$	$C = I\ddot{\theta}$	C is moment of force (aka couple or torque) (sometimes denoted by K) $F = ma = \frac{d}{dt}(mv)$; ie the rate of change of (linear) momentum; $C = I\ddot{\theta} = \frac{d}{dt}(I\dot{\theta})$; ie the rate of change of angular momentum
$\int F\,ds$	$\int C\,d\theta$	work done
$\int F\,dt = [mv]$	$\int C\,dt = [I\dot{\theta}]$	$\int C\,dt$ is the impulse of a torque; the square brackets denote "change in"

Notes

(i) The term 'couple' implies two equal but opposite forces applied to an object, causing it to rotate (without any translation). But it is sometimes used to represent the combined effect of any number of forces on an object that is rotating; ie it is the total moment of the forces. (Such a system can be reduced to an equivalent two force situation.)

(ii) Note that the angular momentum of a particle is:

$I\dot{\theta} = (mr^2)\left(\frac{v}{r}\right) = (mv)r$ ("moment of momentum" is an alternative term for angular momentum)

(iii) Work-Energy equation: $\int C\, d\theta = \left[\frac{1}{2}I(\dot{\theta})^2\right]$

(where the square brackets represent "change in")

ie total work done by moments = change in rotational kinetic energy

Derivation: $\int C\, d\theta = \int I\ddot{\theta}\, d\theta = \int I\ddot{\theta}\frac{d\theta}{dt}dt = \int I\ddot{\theta}\dot{\theta}dt$

$= \left[\frac{1}{2}I(\dot{\theta})^2\right]$ (since $\frac{d}{dt}\left(\frac{1}{2}I(\dot{\theta})^2\right) = I\dot{\theta}\ddot{\theta}$)

[Special case, where there are no external forces: conservation of energy]

(iv) Impulses

Integrating $C = I\ddot{\theta}$ (where C now denotes the total moment of forces (or torque)), we obtain:

$\int C\, dt = [I\dot{\theta}]$

In words: the total impulse of a torque about a given axis of rotation equals the change in angular momentum of the body about the same axis.