Turning points (3 pages; 7/4/20)

See also: "Points of inflexion "

(1) The turning point of a quadratic is midway between its roots.

(2) Note that a turning point can occur when $\frac{d^2y}{dx^2} = 0$ (eg $y = x^4$), and the pattern of $\frac{dy}{dx}$ about the point may need to be examined, instead of the usual test of whether $\frac{d^2y}{dx^2}$ is positive or negative.

(3) A necessary and sufficient condition for a turning point is that the first non-vanishing derivative must be even (with order ≥ 2).

[See Appendix for a sketch of a proof of this.]

The sign of this derivative then determines whether it is a maximum or minimum. Thus, in the case of $y = x^4$ at $x = 0$,

$\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$ & $\frac{d^4y}{dx^4} = 24$

(4) Note that a maximum/minimum can occur without $\frac{dy}{dx} = 0$, if the domain of the function is limited, and the greatest/lowest value occurs at the boundary.

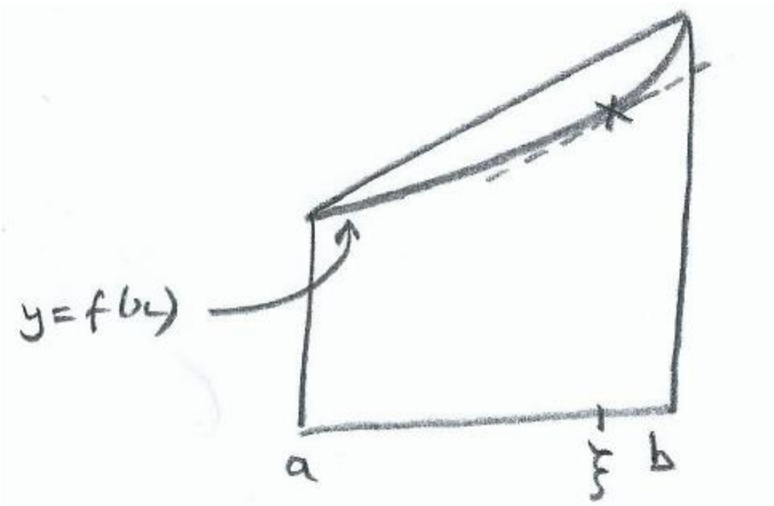
(5) A polynomial function of the form $(x-a)^{2m}g(x)$, where $m > 0$, has a turning point at $(a, 0)$.

Appendix: A necessary and sufficient condition for a turning point is that the first non-vanishing derivative must be even (with order ≥ 2).

Sketch of proof

(A) The Mean Value theorem

This states that "If $f(x)$ has a derivative for all values of x in the interval (a, b), then there is a value ξ of x between a and b, such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$ "



The diagram demonstrates this: the gradient of the curve at

$x = \xi$, ie $f'(\xi)$ equals the gradient of the line (ie $\frac{f(b)-f(a)}{b-a}$).

The theorem can be written in the form

$f(x + h) = f(x) + hf'(x + \theta h)$, where $0 < \theta < 1$ (1)

(with x replacing a and $h = b - a$)

(B) The General Mean Value theorem

It can be shown that (1) extends to

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots \frac{h^n}{n!}f^{(n)}(x+\theta h), \quad (2)$$

where $0 < \theta < 1$

[See, for example, "A course of Pure Mathematics" by G H Hardy (CUP 1933)]

(C) If the first $n-1$ derivatives of $f(x)$ are zero, then

$(2) \Rightarrow f(x+h) - f(x) = \frac{h^n}{n!}f^{(n)}(x+\theta h)$

If n is even, then $f(x+h) - f(x) > 0$ if $f^{(n)}(x+\theta h) > 0$

This is true for positive and negative h, so that there is a local minimum at x.

Also (again with even n), $f(x+h) - f(x) < 0$ if $f^{(n)}(x+\theta h) < 0$, and then there is a local maximum at x.

If instead n is odd, then the sign of $\frac{h^n}{n!}f^{(n)}(x+\theta h)$ will change as h changes from negative to positive, so that there will not be a turning point.

Thus a necessary and sufficient condition for a turning point is that the first non-vanishing derivative must be even (with order ≥ 2).