Vector Triple Products (4 pages; 4/8/18)

(1) Scalar triple product (aka triple scalar product)

$\underline{a}.(\underline{b} \times \underline{c})$ or just $\underline{a}.\underline{b} \times \underline{c}$, as $(\underline{a}.\underline{b}) \times \underline{c}$ is not possible

$$\underline{b} \times \underline{c} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{vmatrix} \underline{i} & b_1 & c_1 \\ \underline{j} & b_2 & c_2 \\ \underline{k} & b_3 & c_3 \end{vmatrix}$$

$= (b_2c_3 - b_3c_2)\underline{i} + (b_3c_1 - b_1c_3)\underline{j} + (b_1c_2 - b_2c_1)\underline{k}$

$\underline{a}.(\underline{b} \times \underline{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (or $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$)

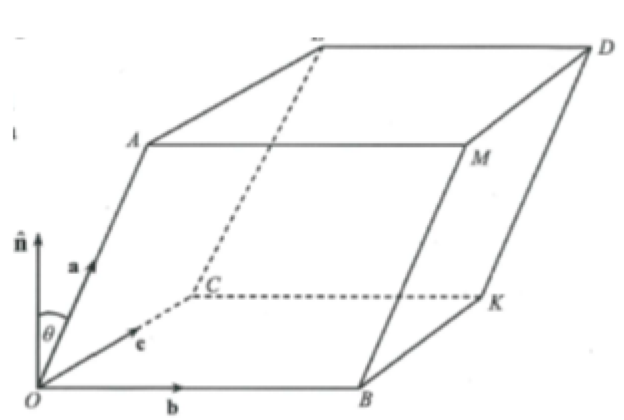
(2) Volumes

The scalar triple product can be used to find the volume of a solid, where it is of the form khA

where k is a number (eg 1, $\frac{1}{3}$ etc), h = height,

and A = area of base

For a parallelepiped (a squashed cuboid, as in the diagram below),



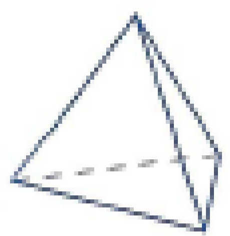
volume $= hA$, where $h = |\underline{a}|cos\theta$ and $A = |\underline{b} \times \underline{c}|$

so that volume $= |\underline{a}|cos\theta|\underline{b} \times \underline{c}| = \underline{a}.(\underline{b} \times \underline{c})$

To allow for the case where $\underline{a}$, $\underline{b}$ & $\underline{c}$ are not in an 'anti-clockwise' order, we can write: volume $= |\underline{a}.(\underline{b} \times \underline{c})|$

For a tetrahedron (ie triangle-based pyramid),

Volume = $\frac{1}{3}$ perp. height × area of base


tetrahedron

Referring to the diagram, if $\underline{b}$ and $\underline{c}$ are vectors along the sides of the base, from one corner, and $\underline{a}$ is the vector from that corner, along a sloping side,

Volume $= |\frac{1}{3}|\underline{a}|cos\theta\left(\frac{1}{2}|\underline{b} \times \underline{c}|\right)| = \frac{1}{6}|\underline{a}.(\underline{b} \times \underline{c})|$

For a square-based pyramid,

Volume $= |\frac{1}{3}|\underline{a}|cos\theta(|\underline{b}\times\underline{c}|)| = \frac{1}{3}|\underline{a}.(\underline{b}\times\underline{c})|$

Example: Find the volume of the tetrahedron with corners

A (0,0,0) B (1,0,0) C (1,2,0) D (1,2,3)

Solution

Volume $= \frac{1}{6}\left|\overrightarrow{AB}.\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right|$

$\overrightarrow{AB} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}1\\2\\0\end{pmatrix}$ $\overrightarrow{AD} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

$\overrightarrow{AB}.\left(\overrightarrow{AC}\times\overrightarrow{AD}\right) = \begin{vmatrix}1 & 1 & 1\\0 & 2 & 2\\0 & 0 & 3\end{vmatrix} = 6$

Hence volume $= \frac{1}{6}|6| = 1$

(3) Scalar triple product results

(i) $\underline{a}.(k\underline{a}\times\underline{b}) = 0$ (as $k\underline{a}\times\underline{b}$ is perpendicular to $\underline{a}$)

(ii) $\underline{a}.(\underline{b}\times\underline{c}) = \underline{b}.(\underline{c}\times\underline{a}) = \underline{c}.(\underline{a}\times b)$ ['cyclic interchange']

(because a determinant is unchanged if its columns (or rows) are interchanged cyclicly; also each expression represents the volume of the same parallelepiped)

(iii) $\underline{a}\times\underline{b}.\underline{c} = \underline{a}.\underline{b}\times\underline{c}$

Proof: LHS $= \underline{c}.\underline{a}\times\underline{b}$ = RHS, from (ii)

LHS = $(a \times b).c = c.(a \times b) = a.(b \times c)$, by cyclic interchange

(iv) $\underline{a}.\left(\underline{b} \times \underline{c}\right) = 0 \Rightarrow$ volume of parallelepiped is zero

$\Rightarrow \underline{a}$, $\underline{b}$ and $\underline{c}$ are coplanar (ie lie in the same plane); assuming that they are non-zero

(This can be used to show that 4 points are coplanar, if they represent the points O, A, B & C in the diagram (where O can now be any point), so that $\underline{a} = \overrightarrow{OA}$ etc.)

This also means that $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly dependent;

ie $\underline{a} = \lambda\underline{b} + \mu\underline{c}$

(4) Vector triple product

The vector triple product $\underline{a} \times (\underline{b} \times \underline{c})$ is not as useful as the scalar triple product.

(i) $\underline{a} \times (\underline{b} \times \underline{c})$ is not necessarily the same as $(\underline{a} \times \underline{b}) \times \underline{c}$

(ii) $\underline{a} \times \left(\underline{b} \times \underline{c}\right) = \left(\underline{a}.\underline{c}\right)\underline{b} - \left(\underline{a}.\underline{b}\right)\underline{c}$

[This can be proved (fairly laboriously) by writing $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ etc, and expanding both sides.]