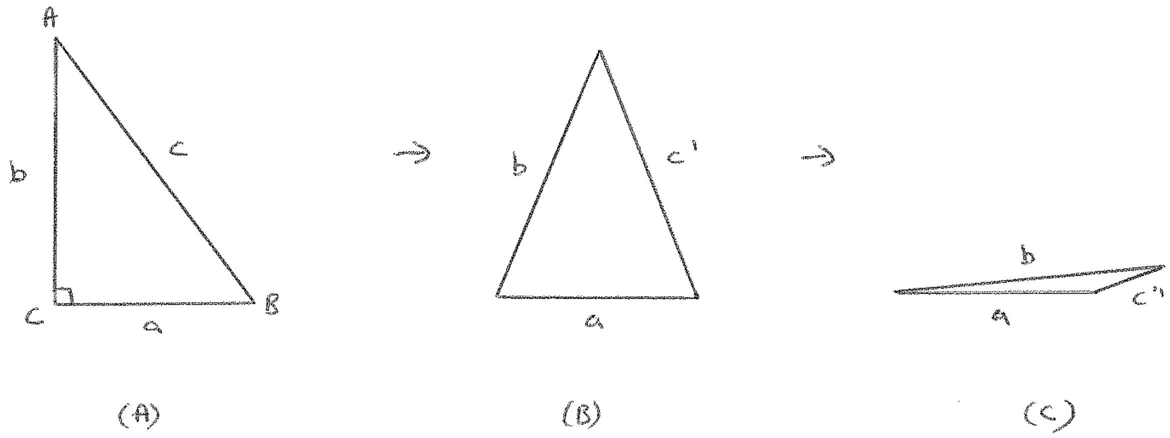
Trigonometry – Sine & Cosine Rules (5 pages; 15/4/21)

(1) Limiting cases of the Cosine rule



Start with a right-angled triangle with sides a, b & c (as in diagram A), such that c is the hypotenuse and side b is longer than side a.

This then gives $c^2 = a^2 + b^2$, by Pythagoras' Theorem.

If the angle C is reduced from being a right-angle, such that the lengths a & b remain the same, then the side c is also reduced (to c' in diagram B).

If C is reduced to zero, then c becomes $c'' = b - a$ (diagram C shows the position as C approaches zero).

If we now consider the adjustment of $-2abcosC$ to the formula for c^2 , we can see that it has the following necessary properties:

(i) it is zero if $C = 90°$

(ii) the adjustment becomes a larger negative value as C reduces

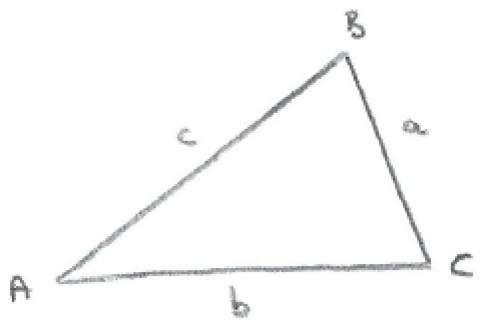
(iii) the adjustment is symmetrical in a & b

(iv) when $C = 0°$, the adjustment becomes $-2ab$, giving

$c^2 = a^2 + b^2 - 2ab = (b-a)^2$, so that $c = b - a$ (if $b > a$)

(2) Applying the Sine & Cosine rules

As $sin\theta = sin\ (180° - \theta)$, care should be taken when using the Sine rule to find an angle in a triangle that is close to 90°. The problem can sometimes be avoided by finding other angles first and subtracting from 180°. There is never any risk with using the Cosine rule, if that is feasible for the given problem.



For the triangle above, the following combinations of sides and angles will always enable the other sides and angles to be determined uniquely (ie any two triangles thus created will be congruent, with a reflection in the plane of the paper being allowed):

(I) a, b & c known

(II) A, B (and hence C) & a (eg) known

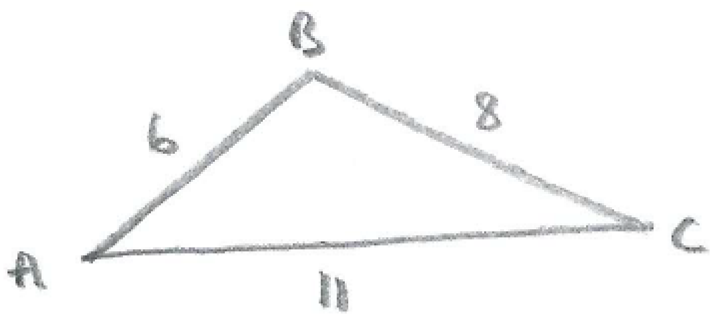
(III) a, b & C $(or\ b, c$ & $A\ etc)$ known

For the following case, there will be two solutions if A is acute, $a < b$ and $B \neq 90°$:

(IV) a, b, A $(or\ B)$ known

For case (I), where a, b & c are known, the following strategy can be applied for finding the angles:

Apply the Cosine rule to find the angle opposite the largest side (or one of the largest sides, in the case of an isosceles triangle). [The reason for this will become clear from the example below.] Then apply either the Cosine or Sine rules to find another angle (the Cosine rule avoids having to take the sine of a rounded angle). Avoid using the Sine rule to find an angle that could be greater than 90° (giving two possible answers - though only one of them will be correct).



Referring to the above diagram, B (the largest angle) can be determined from the Cosine rule:

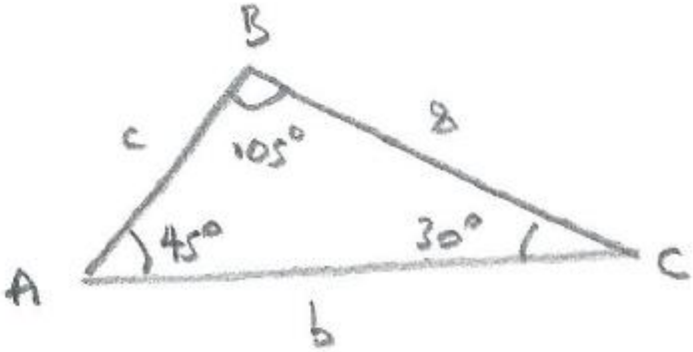
$11^2 = 6^2 + 8^2 - 2(6)(8)cosB$, giving $B = 102.636°$

Then A can be determined either by applying the Cosine rule again, or from the Sine rule:

$\frac{sinA}{8} = \frac{sin102.636°}{11}$

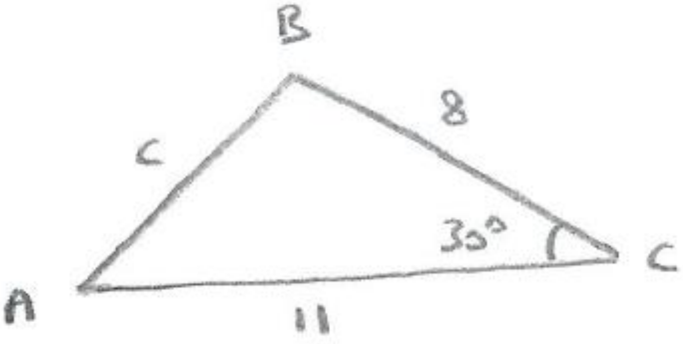
(Had we used the Cosine rule to obtain A (say), rather than B, then it would be unsafe to use the Sine rule to obtain B, since we can't be sure that B is greater than 90°. We could however find C from the Sine rule, and deduce B from A and C; or of course use the Cosine rule again to find either A or B.)

For case (II), where A, B (and hence C) & a are known, simply apply the Sine rule twice (as there is no ambiguity when finding a side from an angle).



Referring to the above diagram, $\frac{b}{sin105°} = \frac{8}{sin45°} = \frac{c}{sin30°}$

For case (III), where a, b & C are known, apply the Cosine rule to find c; then either the Cosine or Sine rules to find another angle (choosing the one opposite the smaller side if using the Sine rule).

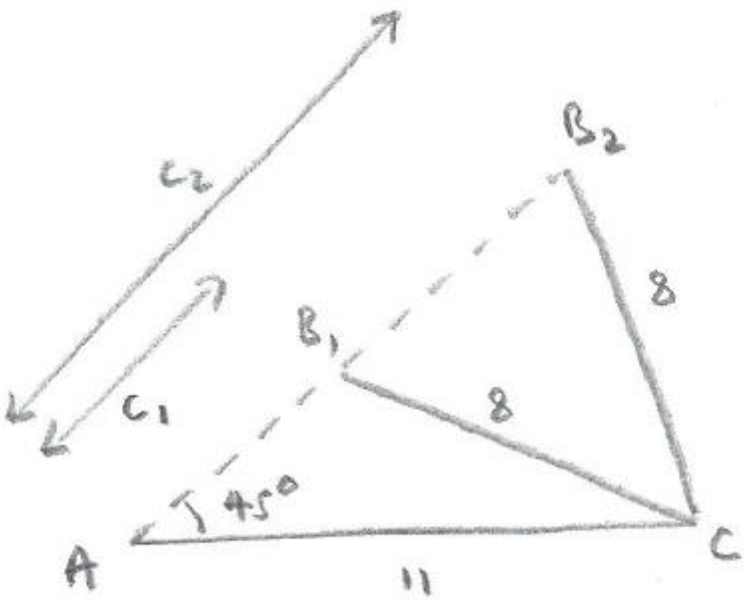


Referring to the above diagram, $c^2 = 11^2 + 8^2 - 2(11)(8)cos30°$,

giving $c = 5.70785$

Then $\frac{sinA}{8} = \frac{sin30°}{5.70785}$ to find A (but not $\frac{sinB}{11} = \frac{sin30°}{5.70785}$, as we don't know whether B is greater or less than 90°). Alternatively, the Cosine rule can be used again to find A or B.

For case (IV), where a, b & A are known, use the Sine rule to find B. If $a < b$, there will be two possible values for B (unless $B = 90°$). If $a \geq b$, B will be acute.



Referring to the above diagram, $\frac{sin45°}{8} = \frac{sinB}{11}$, giving $sinB = 0.97227$

and $B = 76.476°$ or $103.525°$

Alternatively, the Cosine rule could be used to find the two possible values for c:

$8^2 = 11^2 + c^2 - 2(11)c(cos45°)$