Trigonometry - Radians (4 pages; 15/4/21)

(1) Definition

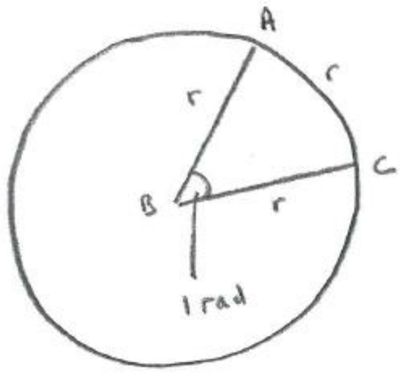

Fig. 1

In Fig. 1, the arc length AC equals the radius of the circle, and the angle ABC is defined to be 1 radian.

The chord AC is just smaller than r. Hence the triangle ABC is a slightly squashed equilateral triangle, and so 1 rad is just less than 60^0 .

The exact size can be determined by proportional reasoning, using the table below; together with other useful facts.

	angle (deg)	angle (rad)	arc length	area of sector
1	a	1	r	
2	360	b	$2\pi r$	πr^2
3	c	θ	d	e
4	ϕ	f		

Line 1 is based on the definition of the radian.

Line 2 is based on what we know about the circumference and area of a circle.

From the arc length column, we see that line 2 is 2π times line 1.

Thus $a = \frac{360}{2\pi} = \frac{180}{\pi} = 57.3°\ (3sf)$; ie 1 radian is approx. 57.3°

And $b = 2\pi(1) = 2\pi$; ie there are 2π radians in a circle.

Then, as line 3 is θ times line 1, $c = \theta a = \theta\left(\frac{360}{2\pi}\right)$ or $\theta\left(\frac{180}{\pi}\right)$; ie we can convert from radians to degrees by multiplying by $\frac{360}{2\pi}$ (or $\frac{180}{\pi}$).

Also, $d = \theta r$.

Then noting, from the arc length column, that line 3 is $\frac{\theta r}{2\pi r}$ times line 2,

$e = \left(\frac{\theta r}{2\pi r}\right)(\pi r^2) = \frac{1}{2}\theta r^2$, which is the area of a sector with an angle of θ rad.

[As an aid to memory, the triangle ABC in Fig. 1 has area $\frac{1}{2}r^2 sin\theta$, and as $sin\theta \to \theta\ as\ \theta \to 0$ [see "Small Angle Approximations" in Part 2], this area tends to $\frac{1}{2}r^2\theta$]

Finally, $f = (1)\left(\frac{\phi}{a}\right) = \left(\frac{2\pi}{360}\right)\phi$, to give the radian equivalent of an angle in degrees.

[As the angle in degrees is much larger than the corresponding angle in radians, there should never be any doubt whether to multiply or divide by $\frac{360}{2\pi}$ when switching between degrees and radians.]

(2) Why are radians preferred to degrees?

(i) The key point is that $sin\theta \approx \theta$ for small θ measured in radians, but, if ϕ is measured in degrees, then

$$sin\phi = sin\theta \approx \theta = \left(\frac{\pi}{180}\right)\phi$$

So $\lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$ when θ is measured in radians

and $\lim_{\phi \to 0} \frac{sin\phi}{\phi} = \frac{\pi}{180}$ when ϕ is measured in degrees

(ii) Consider the graph of $sin\phi$, where ϕ is measured in degrees, and compare it with the graph of $sin\theta$, where θ is measured in radians. $sin\phi$ increases from 0 to 1 as ϕ increases from 0° to 90°, whereas $sin\theta$ increases from 0 to 1 as θ increases from 0 to $\frac{\pi}{2}$.

Thus the graph of $sin\phi$ is more stretched out than that of $sin\theta$, with a much smaller gradient (except when $cos\theta = cos\phi = 0$).

In particular, at the Origin, $y = sin\theta$ tends to $y = \theta$ only when θ is measured in radians.

(iii) When measuring angles in radians,

$$\frac{d}{d\theta}sin\theta = \lim_{h \to 0} \frac{\sin(\theta+h) - sin\theta}{h} = \lim_{h \to 0} \frac{sin\theta\cos(h) + cos\theta\sin(h) - sin\theta}{h}$$

As $\cos(h) \to 1$ as $h \to 0$, and $\frac{\sin(h)}{h} \to 1$, as we are measuring our angles in radians, $\frac{d}{d\theta}sin\theta = cos\theta$

But when measuring angles in degrees,

$\frac{d}{d\phi} sin\phi = \lim_{h \to 0} \frac{sin\phi\cos(h) + cos\phi\sin(h) - sin\phi}{h}$ again,

and it is still true that $\cos(h) \to 1$ as $h \to 0$, but now $\frac{\sin(h)}{h} \to \frac{\pi}{180}$

so that $\frac{d}{d\phi} sin\phi = \frac{\pi}{180} cos\phi$ (where ϕ is measured in degrees)

Note that, strictly speaking, it is not enough for ϕ to have a value which happens to be the number of degrees: the cosine (or sine) function itself is different, depending on whether the angle is measured in degrees or radians. To be clear, we could use the notation $sin_{deg}\phi$ and $sin_{rad}\theta$ (as in the next part).

(iv) Alternative derivation:

If ϕ is the angle in degrees, and θ is the angle in radians, so that $\phi = \left(\frac{180}{\pi}\right)\theta$, then

$$\frac{d}{d\phi} sin_{deg}\phi = \frac{d}{d\phi} sin_{rad}\theta = \left[\frac{d}{d\theta} sin_{rad}\theta\right]\frac{d\theta}{d\phi} = (cos_{rad}\theta)\left(\frac{\pi}{180}\right)$$

$$= (cos_{deg}\phi)\left(\frac{\pi}{180}\right)$$

Or: $\frac{d}{d\phi} sin_{deg}\phi = \frac{d}{d\phi} sin_{rad}\theta = \frac{d}{d\phi} sin_{rad}[\phi\left(\frac{\pi}{180}\right)]$

$$= \left(\frac{\pi}{180}\right) cos_{rad}\left[\phi\left(\frac{\pi}{180}\right)\right] = \left(\frac{\pi}{180}\right) cos_{rad}\theta = \left(\frac{\pi}{180}\right) cos_{deg}\phi$$