

Trigonometry - Part 2 (15 pages; 22/3/20)

(8) Solution of simple equations - for a range of angles

Example: To solve $\sin\left(2\theta - \frac{\pi}{6}\right) = 0.5$ for $0 < \theta \leq 2\pi$,

let $\phi = 2\theta - \frac{\pi}{6}$, so that $-\frac{\pi}{6} < \phi \leq 4\pi - \frac{\pi}{6}$

Then $\phi = \frac{\pi}{6}$, $\pi - \frac{\pi}{6}$, and also $\frac{\pi}{6} + 2\pi$ & $\pi - \frac{\pi}{6} + 2\pi$

and values for θ are obtained from $\theta = \frac{1}{2}\left(\phi + \frac{\pi}{6}\right)$

[When $\cos\phi = 0.5$, $\phi = \frac{\pi}{3}$, $2\pi - \frac{\pi}{3}$, and multiples of 2π can be added or subtracted.]

(9) General solution of simple equations

The approach above (for a limited range of θ) can be applied, to obtain two ('base') solutions in one cycle (except for $\sin\theta = 1$ or -1). Then the general solutions can be derived as follows:

(i) For eg $\sin\theta = 0.5$, the base solutions are $\theta = \frac{\pi}{6}$, $\pi - \frac{\pi}{6}$

A more concise alternative to $\frac{\pi}{6} + 2k\pi$ or $\frac{5\pi}{6} + 2k\pi$ is to note that the solutions lie alternately $\frac{\pi}{6}$ ahead of and behind the

multiples of π , so that we can write $\theta = n\pi + (-1)^n \left(\frac{\pi}{6}\right)$

So the general solution for $y = \sin x$ is $x = n\pi + (-1)^n \arcsin y$

(ii) For eg $\cos\theta = 0.5$, the base solutions are $\theta = \frac{\pi}{3}$, $2\pi - \frac{\pi}{3}$, or alternatively $\theta = \frac{\pi}{3}$ & $-\frac{\pi}{3}$, from which we obtain the general solution of $\theta = 2k\pi \pm \frac{\pi}{3}$

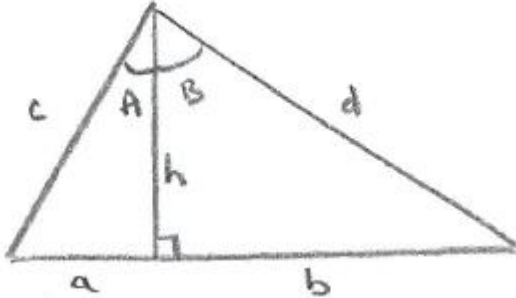
So the general solution for $y = \cos x$ is $x = 2k\pi \pm \arccos y$

(iii) For eg $\tan\theta = 1$, the general solution is just $\theta = \frac{\pi}{4} + k\pi$

So the general solution for $y = \tan x$ is $x = \arctan y + k\pi$

(10) Compound angle formulae

Proof of $\sin(A + B) = \sin A \cos B + \cos A \sin B$:



Referring to the diagram,

the area of the large triangle can be formed in two ways:

$$\frac{1}{2}cd\sin(A + B) = \frac{1}{2}ah + \frac{1}{2}bh$$

$$\Rightarrow \sin(A + B) = \frac{a}{c} \cdot \frac{h}{d} + \frac{b}{d} \cdot \frac{h}{c} = \sin A \cos B + \sin B \cos A$$

The other compound angle formulae can be derived from the formula for $\sin(A + B)$, as follows:

$$\sin(A - B) = \sin(A + [-B]) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$= \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \sin(90 - [A + B]) = \sin([90 - A] - B)$$

$$= \sin(90 - A) \cos B - \cos(90 - A) \sin B$$

$$= \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos(A + [-B]) = \cos A \cos(-B) - \sin A \sin(-B)$$

$$= \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad (\text{after dividing top and bottom by } \cos A \cos B)$$

$$\text{and similarly for } \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Proof using matrices

Rotation of θ followed by rotation of ϕ

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

$$\text{Also } \begin{pmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

$$\text{Hence } \cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$$

$$\& \sin(\theta + \phi) = \cos\theta\sin\phi + \sin\theta\cos\phi \quad \text{or} \quad \sin\theta\cos\phi + \cos\theta\sin\phi$$

(11) $\cos^2\theta$ and $\sin^2\theta$

Using the results $\cos^2\theta + \sin^2\theta = 1$

(by applying Pythagoras to a right-angled triangle with sides $\cos\theta, \sin\theta$ & 1)

and $\cos^2\theta - \sin^2\theta = \cos(2\theta)$ (from the Compound angle formulae) we can derive:

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)) \quad \& \quad \sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

(12) $R\sin(\theta + \alpha)$ and $R\cos(\theta - \alpha)$

The compound angle formulae can be used to write the expression $a\sin\theta + b\cos\theta$ in either of the forms $R\sin(\theta + \alpha)$ or $R\cos(\theta - \alpha)$.

Example

$$\sqrt{3}\sin\theta + \cos\theta = R\sin(\theta + \alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

and equating coefficients of $\sin\theta$ & $\cos\theta$ (given that the two expressions are to be equal for all values of θ),

$$\sqrt{3} = R\cos\alpha \quad \& \quad 1 = R\sin\alpha,$$

$$\text{so that } 3 + 1 = R^2(\cos^2\alpha + \sin^2\alpha) = R^2,$$

$$\text{giving } R = 2 \text{ (for example); and } \tan\alpha = \frac{1}{\sqrt{3}}, \text{ giving } \alpha = \frac{\pi}{6}$$

$$\text{Thus } \sqrt{3}\sin\theta + \cos\theta = 2\sin\left(\theta + \frac{\pi}{6}\right)$$

Alternatively,

$$\sqrt{3}\sin\theta + \cos\theta = R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha,$$

so that $\sqrt{3} = R\sin\alpha$ & $1 = R\cos\alpha$, giving $R = 2$ again,

and $\tan\alpha = \sqrt{3}$, so that $\alpha = \frac{\pi}{3}$

Thus $\sqrt{3}\sin\theta + \cos\theta$ can be written as either

$$2\sin\left(\theta + \frac{\pi}{6}\right) \text{ or } 2\cos\left(\theta - \frac{\pi}{3}\right) \text{ [see note (b) below]}$$

Had we chosen $R\sin(\theta - \alpha)$ instead, then

$$\sqrt{3}\sin\theta + \cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha,$$

giving $\sqrt{3} = R\cos\alpha$ & $1 = -R\sin\alpha$,

so that $R = 2$ and $\tan\alpha = -\frac{1}{\sqrt{3}}$, or $\tan(-\alpha) = \frac{1}{\sqrt{3}}$,

with the same result as before.

Similarly,

$$\sqrt{3}\sin\theta + \cos\theta = R\cos(\theta + \alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha,$$

giving $\sqrt{3} = -R\sin\alpha$ & $1 = R\cos\alpha$,

so that $R = 2$ and $\tan\alpha = -\sqrt{3}$, or $\tan(-\alpha) = \sqrt{3}$, as before

Notes

(a) $R\sin(\theta - \alpha)$ can be chosen, if b is negative; similarly for $R\cos(\theta + \alpha)$ if a is negative.

(b) As $\sin(\theta + \alpha) = \cos(90^\circ - [\theta + \alpha]) = \cos([90^\circ - \alpha] - \theta)$
 $= \cos(\theta - [90^\circ - \alpha])$, one form can be obtained from the other.
 Similarly, $\cos(\theta - \alpha) = \cos(\alpha - \theta) = \sin(90^\circ - [\alpha - \theta])$
 $= \sin(\theta + [90^\circ - \alpha])$

(13) Factor formulae

$$\sin\theta + \sin\phi = 2\sin X \cos Y$$

$$\sin\theta - \sin\phi = 2\cos X \sin Y$$

$$\cos\theta + \cos\phi = 2\cos X \cos Y$$

$$\cos\theta - \cos\phi = -2\sin X \sin Y$$

where $X = \frac{1}{2}(\theta + \phi)$ & $Y = \frac{1}{2}(\theta - \phi)$

Proofs

Let $\theta = X + Y$ & $\phi = X - Y$

Then $\sin\theta + \sin\phi = \sin X \cos Y + \cos X \sin Y + \sin X \cos Y - \cos X \sin Y$
 $= 2\sin X \cos Y$, with X & Y as above; and similarly for the other formulae.

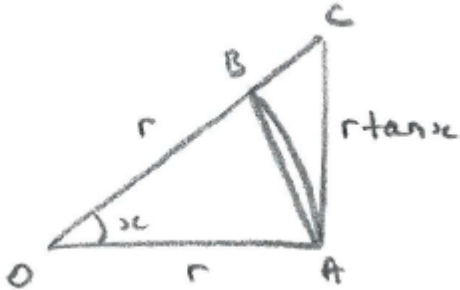
(14) Small angle approximations

These can be derived for $\sin x$ & $\cos x$ from their Maclaurin expansions [See "Maclaurin Series"].

Thus $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \Rightarrow \sin x \approx x$ for small x ,

$$\& \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \Rightarrow \cos x \approx 1 - \frac{x^2}{2} \text{ for small } x$$

Alternative derivation



For $\sin x \approx x$:

$$\triangle OAB < \text{sector } OAB < \triangle OAC,$$

$$\text{so that } \frac{1}{2}r^2 \sin x < \frac{1}{2}r^2 x < \frac{1}{2}r(r \tan x)$$

$$\text{and hence } 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

As $x \rightarrow 0$, $\frac{1}{\cos x} \rightarrow 1$ (from above);

ie we can make $\frac{1}{\cos x}$ as close to 1 as we please,

and then $\frac{x}{\sin x} \approx 1$; ie $\sin x \approx x$ for small x

For $\cos x \approx 1 - \frac{x^2}{2}$:

$$\text{Starting with } \cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta,$$

$$\text{and writing } x = 2\theta, \text{ we have } \cos x = 1 - 2\sin^2 \left(\frac{x}{2}\right)$$

Then, for small x , $\cos x \approx 1 - 2\left(\frac{x}{2}\right)^2 = 1 - \frac{x^2}{2}$

(15) Why are radians preferred to degrees?

(i) The key point is that $\sin\theta \approx \theta$ for small θ measured in radians, but, if ϕ is measured in degrees, then

$$\sin\phi = \sin\theta \approx \theta = \left(\frac{\pi}{180}\right)\phi$$

So $\lim_{\theta \rightarrow 0} \frac{\sin\theta}{\theta} = 1$ when θ is measured in radians

and $\lim_{\phi \rightarrow 0} \frac{\sin\phi}{\phi} = \frac{\pi}{180}$ when ϕ is measured in degrees

(ii) Consider the graph of $\sin\phi$, where ϕ is measured in degrees, and compare it with the graph of $\sin\theta$, where θ is measured in radians. $\sin\phi$ increases from 0 to 1 as ϕ increases from 0° to 90° , whereas $\sin\theta$ increases from 0 to 1 as θ increases from 0 to $\frac{\pi}{2}$.

Thus the graph of $\sin\phi$ is more stretched out than that of $\sin\theta$, with a much smaller gradient (except when $\cos\theta = \cos\phi = 0$).

In particular, at the Origin, $y = \sin\theta$ tends to $y = \theta$ only when θ is measured in radians.

(iii) When measuring angles in radians,

$$\frac{d}{d\theta} \sin\theta = \lim_{h \rightarrow 0} \frac{\sin(\theta+h) - \sin\theta}{h} = \lim_{h \rightarrow 0} \frac{\sin\theta \cos(h) + \cos\theta \sin(h) - \sin\theta}{h}$$

As $\cos(h) \rightarrow 1$ as $h \rightarrow 0$, and $\frac{\sin(h)}{h} \rightarrow 1$, as we are measuring our angles in radians, $\frac{d}{d\theta} \sin\theta = \cos\theta$

But when measuring angles in degrees,

$$\frac{d}{d\phi} \sin\phi = \lim_{h \rightarrow 0} \frac{\sin\phi \cos(h) + \cos\phi \sin(h) - \sin\phi}{h} \text{ again,}$$

and it is still true that $\cos(h) \rightarrow 1$ as $h \rightarrow 0$, but now $\frac{\sin(h)}{h} \rightarrow \frac{\pi}{180}$

so that $\frac{d}{d\phi} \sin\phi = \frac{\pi}{180} \cos\phi$ (where ϕ is measured in degrees)

Note that, strictly speaking, it is not enough for ϕ to have a value which happens to be the number of degrees: the cosine (or sine) function itself is different, depending on whether the angle is measured in degrees or radians. To be clear, we could use the notation $\sin_{deg}\phi$ and $\sin_{rad}\theta$ (as in the next part).

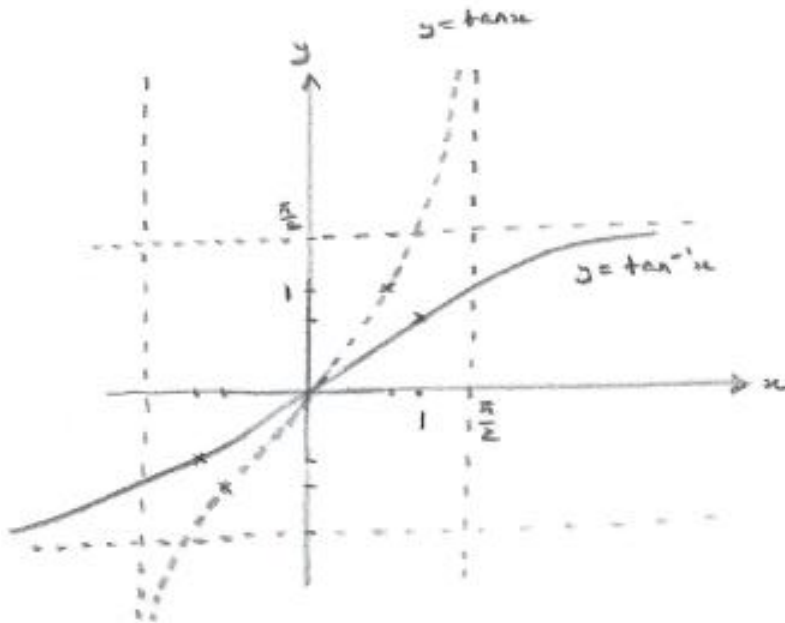
(iv) Alternative derivation:

If ϕ is the angle in degrees, and θ is the angle in radians, so that $\phi = \left(\frac{180}{\pi}\right)\theta$, then

$$\begin{aligned} \frac{d}{d\phi} \sin_{deg}\phi &= \frac{d}{d\phi} \sin_{rad}\theta = \left[\frac{d}{d\theta} \sin_{rad}\theta \right] \frac{d\theta}{d\phi} = (\cos_{rad}\theta) \left(\frac{\pi}{180} \right) \\ &= (\cos_{deg}\phi) \left(\frac{\pi}{180} \right) \end{aligned}$$

$$\text{Or: } \frac{d}{d\phi} \sin_{deg}\phi = \frac{d}{d\phi} \sin_{rad}\theta = \frac{d}{d\phi} \sin_{rad}\left[\phi \left(\frac{\pi}{180}\right)\right]$$

$$= \left(\frac{\pi}{180}\right) \cos_{rad}\left[\phi \left(\frac{\pi}{180}\right)\right] = \left(\frac{\pi}{180}\right) \cos_{rad}\theta = \left(\frac{\pi}{180}\right) \cos_{deg}\phi$$

(16) Inverse Trig. Functions**(A) $y = \tan^{-1}x$ (or $\arctan x$)**

(i) The scales have to be in radians in order for these graphs to be reflections of each other in $y = x$.

(ii) In order to establish the gradient of $y = \tan x$ at the Origin:

$$\frac{d}{dx} (\tan x) = \sec^2 x = 1 \text{ when } x = 0$$

(this assumes that the angle is in radians)

[The above graph hasn't been drawn that well: the gradients of both $y = \tan x$ and $y = \tan^{-1} x$ are intended to be 1 at the Origin.]

(iii) In order for $y = \tan^{-1} x$ to be a 1-1 mapping (and therefore a function), the domain of $y = \tan x$ has to be limited to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$$(iv) \frac{d}{dx} \tan^{-1}x$$

$$\text{Let } y = \tan^{-1}x$$

Then $\tan y = x$ and $\sec^2 y \frac{dy}{dx} = 1$ (differentiating implicitly wrt x) [alternatively, differentiate wrt y , to give $\sec^2 y = \frac{dx}{dy}$ and take the reciprocal]

$$\text{And as } \sec^2 y = \tan^2 y + 1 = x^2 + 1, \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\text{Also } \int \frac{1}{1+x^2} dx = \tan^{-1}x + c$$

[See also "Integration methods".]

(v) Features of $\frac{dy}{dx}$ (in agreement with graph):

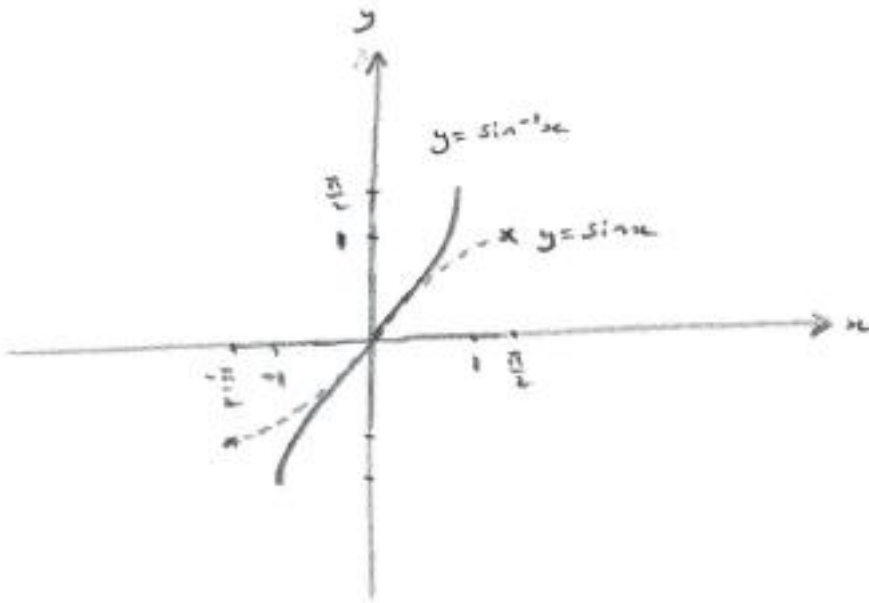
$$(a) \frac{dy}{dx} \rightarrow 0 \text{ as } x \rightarrow \pm \infty$$

$$(b) \frac{dy}{dx} \text{ is always positive}$$

$$(c) \frac{dy}{dx} = 1 \text{ when } x = 0$$

$$(vi) \tan x = y \rightarrow x = \arctan y + n\pi$$

(B) $y = \sin^{-1}x$ (or \arcsinx)



(i) In order for $y = \sin^{-1}x$ to be a 1-1 mapping (and therefore a function), the domain of $y = \sin x$ has to be limited to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(ii) $\frac{d}{dx} (\sin x) = 1$ when $x = 0$

(iii) $\sin x = y \rightarrow x = (\arcsin y \text{ or } \pi - \arcsin y) + 2n\pi$

or $x = n\pi + (-1)^n \arcsin y$

(iv) $\frac{d}{dx} \sin^{-1}x$

Let $y = \sin^{-1}x$, so that $\sin y = x$ and $\cos y \frac{dy}{dx} = 1$

Hence $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$,

taking the positive root, as y is restricted to the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$,
 where $\cos y > 0$ (assuming that $\frac{dy}{dx}$ is defined, so that $\cos y \neq 0$)

(also $\frac{dy}{dx} > 0$ from the graph).

Also $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x$ (see "Integration Methods").

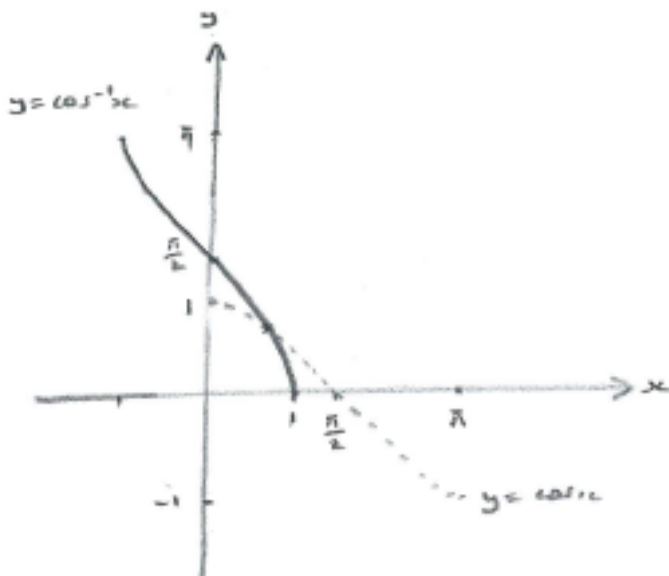
(v) Features of $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

(a) $\frac{dy}{dx} \rightarrow \infty$ as $x \rightarrow \pm 1$

(b) $\frac{dy}{dx}$ is always positive

(c) $\frac{dy}{dx}$ is undefined when $x \leq -1$ or $x \geq 1$

(C) $y = \cos^{-1}x$ (or $\arccos x$)



(i) In order for $y = \cos^{-1}x$ to be a 1-1 mapping (and therefore a function), the domain of $y = \cos x$ has to be limited to $[0, \pi]$.

$$(ii) \cos x = y \rightarrow x = (\arccos y \text{ or } 2\pi - \arccos y) + 2n\pi$$

$$\text{or } \pm \arccos y + 2n\pi$$

$$(iii) \frac{d}{dx} \cos^{-1}x$$

$$\text{Let } y = \cos^{-1}x, \text{ so that } \cos y = x \text{ and } -\sin y \frac{dy}{dx} = 1$$

$$\text{Hence } \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$$

taking the positive root, as y is restricted to the range $[0, \pi]$,

when $\sin y > 0$ (assuming that $\frac{dy}{dx}$ is defined, so that $\sin y \neq 0$)

(also $\frac{dy}{dx} < 0$ from the graph)

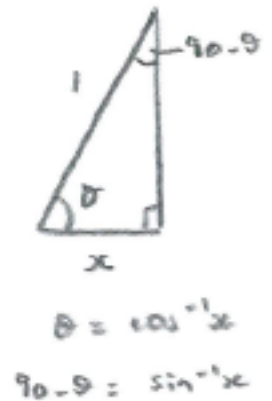
Also $\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1}x$ (see "Integration Methods").

Note: $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$ (see diagram),

$$\text{so that } -\cos^{-1}x + c = \sin^{-1}x - \frac{\pi}{2} + c$$

Thus the two alternative expressions for $\int \frac{1}{\sqrt{1-x^2}} dx$

($\sin^{-1}x$ & $-\cos^{-1}x$) differ by a constant.



(iv) $y = \cos^{-1}x$ is the reflection of $y = \sin^{-1}x$ in $y = \frac{\pi}{4}$, since

$$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$$

(17) $t = \tan\left(\frac{x}{2}\right)$ substitution

See separate note