

## Trigonometry - Exercises: Part 2 (Sol'ns) (4 pages; 6/2/20)

(1\*\*) If the point  $(x, y)$  is rotated (anti-clockwise) about the Origin by an infinitesimal angle  $\delta\theta$  (radians), show that the changes in the coordinates are given by:  $\delta x = -y\delta\theta$  &  $\delta y = x\delta\theta$

### Solution

$$\delta x = r\cos(\theta + \delta\theta) - r\cos\theta$$

$$= r\cos\theta\cos\delta\theta - r\sin\theta\sin\delta\theta - r\cos\theta$$

$$= r\cos\theta\left(1 - \frac{1}{2}(\delta\theta)^2 + \dots\right) - r\sin\theta\left(\delta\theta - \frac{1}{6}(\delta\theta)^3 + \dots\right) - r\cos\theta$$

$$= -r\sin\theta(\delta\theta) \quad \text{to 1st order in } \delta\theta$$

$$= -y\delta\theta, \text{ as required}$$

$$\text{And } \delta y = r\sin(\theta + \delta\theta) - r\sin\theta$$

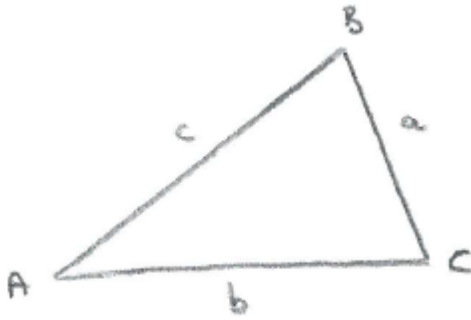
$$= r\sin\theta\cos\delta\theta + r\cos\theta\sin\delta\theta - r\sin\theta$$

$$= r\sin\theta\left(1 - \frac{1}{2}(\delta\theta)^2 + \dots\right) + r\cos\theta\left(\delta\theta - \frac{1}{6}(\delta\theta)^3 + \dots\right) - r\sin\theta$$

$$= r\cos\theta(\delta\theta) \quad \text{to 1st order in } \delta\theta$$

$$= x\delta\theta, \text{ as required}$$

(2\*\*) Denote the sides of a triangle by  $a, b$  &  $c$ , and the angles (opposite these sides respectively) by  $A, B$  &  $C$ , as in the diagram below.



(i) What combinations of sides and angles will always enable the other sides and angles to be determined uniquely (ie any two triangles thus created will be congruent, so that a reflection in the plane of the paper is allowed)? What combination gives rise to two possibilities in some cases?

(ii) When finding missing lengths and angles, what ambiguous situation can arise (apart from the 2 solutions in IV), and how can it be avoided?

### Solution

(i)

(I)  $a, b$  &  $c$  known  $\Rightarrow$  unique solution

(II)  $A, B$  (and hence  $C$ ) &  $a$  (eg)  $\Rightarrow$  unique solution

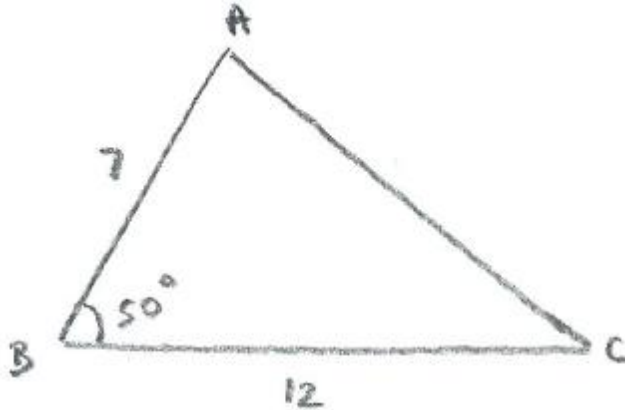
(III)  $a, b$  &  $C$  (or  $b, c$  &  $A$  etc)  $\Rightarrow$  unique solution

(IV)  $a, b, A$  (or  $B$ )  $\Rightarrow$  2 solutions in some cases (if  $A$  is acute,  $a < b$  and  $B \neq 90^\circ$ )

(ii) If we attempt to find an angle that is close to  $90^\circ$  using the Sine rule, then it will not be clear whether the correct angle is  $\theta$  or  $180 - \theta$ .

To avoid this problem we can either use the Cosine rule, or apply the Sine rule only to angles that are clearly acute, and deduce the remaining angle by subtraction from  $180^\circ$ . Note that the angle opposite the longest side is the only one that can be obtuse.

(3\*\*) For the triangle below, what is the best strategy for finding angle  $A$ ?



### Solution

Although there is only one way of drawing the triangle, we want to avoid using  $\sin A$ , as  $A$  is close to  $90^\circ$

(and  $\sin x = \sin(180^\circ - x)$ )

But we could find  $C$  instead, and subtract  $B + C$  from  $180^\circ$ .

We can use the Cosine rule to find  $AC$ , and then the Sine rule to find  $C$ , and hence  $A$ .

Alternatively, having found  $AC$ , we could use the Cosine rule again to find  $A$  (there is never any ambiguity when using the Cosine rule).

(4\*\*) Assuming that  $\sin^2 \theta + \cos^2 \theta = 1$ , but without using any compound angle results, show that  $\sin \theta \cos \theta \leq \frac{1}{2}$

### Solution

$$(\sin \theta - \cos \theta)^2 \geq 0 \Rightarrow \sin^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta \geq 0$$

$$\Rightarrow 1 \geq 2\sin\theta\cos\theta \Rightarrow \sin\theta\cos\theta \leq \frac{1}{2}$$

(5\*\*) Show that  $\frac{d}{d\phi} \sin\phi = \frac{\pi}{180} \cos\phi$ , when  $\phi$  is measured in degrees.

### Solution

If  $\phi$  is the angle in degrees, and  $\theta$  is the angle in radians, so that  $\phi = \left(\frac{180}{\pi}\right)\theta$ , then

$$\begin{aligned} \frac{d}{d\phi} \sin_{deg}\phi &= \frac{d}{d\phi} \sin_{rad}\theta = \left[ \frac{d}{d\theta} \sin_{rad}\theta \right] \frac{d\theta}{d\phi} = (\cos_{rad}\theta) \left( \frac{\pi}{180} \right) \\ &= (\cos_{deg}\phi) \left( \frac{\pi}{180} \right) \end{aligned}$$

[See "Trigonometry - Part 2" for alternative derivations, and related discussion.]