Trigonometry - Exercises: Part 1 (Sol'ns) (11 pages; 6/2/20)

(1***) Solve the equation $sinx - cosx = 0.5,\ for\ 0° < x < 360°$

Solution

Method 1

Write $sinx - cosx = Rsin(x - \alpha) = R(sinxcos\alpha - cosxsin\alpha)$,

so that $Rcos\alpha = 1$ & $Rsin\alpha = 1$,

and hence $R^2(cos^2\alpha + sin^2\alpha) = 2$, so that $R = \sqrt{2}$

Also $tan\alpha = 1$, so that $\alpha = 45°$ (for example).

Thus the original equation becomes $\sqrt{2}sin(x - 45°) = 0.5$

Then let $u = x - 45°$, so that $-45° < u < 315°$

$sinu = \frac{1}{2\sqrt{2}} \Rightarrow u = 20.70481\ or\ 180 - 20.70481$

(and there are no other solutions within the range for u)

So $x = u + 45° = 65.7°\ or\ 204.3°$ (1dp)

Method 2

$sinx - cosx = 0.5 \Rightarrow tanx - 1 = 0.5secx$

$\Rightarrow (tanx - 1)^2 = \frac{sec^2x}{4}$,

if we exclude solutions of $tanx - 1 = -0.5secx$

$\Rightarrow 4(tan^2x - 2tanx + 1) = 1 + tan^2x$

$\Rightarrow 3tan^2x - 8tanx + 3 = 0$

$\Rightarrow tanx = \frac{8\pm\sqrt{28}}{6} = \frac{1}{3}(4 \pm \sqrt{7}) = 2.21525\ or\ 0.45142$

$\Rightarrow x = 65.7°\ or\ 24.3°$,

as well as $65.7 + 180 = 245.7°$ and $24.3 + 180 = 204.3°$

But $24.3°\ \ and\ \ 245.7°$ are solutions of $tanx - 1 = -0.5secx$ and can therefore be excluded.

Thus the solutions are $x = 65.7°\ or\ 204.3°$

Method 3

$sinx - cosx = 0.5 \Rightarrow sin^2x = (cosx + 0.5)^2$

but this will include solutions of $-sinx - cosx = 0.5$, which will need to be removed

$\Rightarrow 1 - cos^2x = cos^2x + cosx + \frac{1}{4}$

$\Rightarrow 2cos^2x + cosx - \frac{3}{4} = 0$

$\Rightarrow 8cos^2x + 4cosx - 3 = 0$

$\Rightarrow cosx = \frac{-4\pm\sqrt{16+96}}{16} = \frac{-1\pm\sqrt{7}}{4} = -0.91144$ or 0.41144

$\Rightarrow x = 155.7°, 360 - 155.7 = 204.3°, 65.7°$

$or\ 360 - 65.7 = 294.3°$

The only solutions of the required equation are

$x = 65.7°\ and\ 204.3°$

(the other two are found to be solutions of $-sinx - cosx = 0.5$)

Method 4

$t = tan\left(\frac{x}{2}\right) \Rightarrow cosx = \frac{1-t^2}{1+t^2}$ & $sinx = \frac{2t}{1+t^2}$ (standard results - see "Trigonometry - Part 2")

Then, substituting into our equation:

$\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = \frac{1}{2}$

$\Rightarrow 2\{2t - (1-t^2)\} = 1 + t^2 \quad \Rightarrow t^2 + 4t - 3 = 0$

$\Rightarrow t = \frac{-4 \pm \sqrt{28}}{2} = -2 \pm \sqrt{7} = 0.64575 \; or - 4.64575$

$\Rightarrow \frac{x}{2} = 32.852° \; or - 77.852° + 180°$

(these are the only values between $0° \; and \; 180°$, which is the permissible range for $\frac{x}{2}$)

and hence $x = 65.7° \; or \; 204.3°$ (1dp)

(2**) Given that $cos^5\theta = \frac{1}{16}(cos5\theta + 5cos3\theta + 10cos\theta)$ and

$cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$,

find expressions for $sin^5\theta$ and $sin^6\theta$

Solution

$sin^5\theta = cos^5\left(\frac{\pi}{2} - \theta\right)$

$= \frac{1}{16}(\cos[5\left(\frac{\pi}{2} - \theta\right)] + 5\cos[3(\frac{\pi}{2} - \theta)] + 10\cos(\frac{\pi}{2} - \theta))$

$= \frac{1}{16}(\cos[\frac{\pi}{2} - 5\theta] + 5\cos[-\frac{\pi}{2} - 3\theta] + 10\sin\theta)$

$= \frac{1}{16}(\sin 5\theta + 5\cos(\frac{\pi}{2} + 3\theta) + 10\sin\theta)$

$= \frac{1}{16}(\sin 5\theta + 5\cos(\frac{\pi}{2} - [-3\theta]) + 10\sin\theta)$

$= \frac{1}{16}(sin5\theta + 5\sin(-3\theta) + 10sin\theta)$

$= \frac{1}{16}(sin5\theta - 5\sin 3\theta + 10sin\theta)$

And $sin^6\theta = cos^6\left(\frac{\pi}{2} - \theta\right)$

$= \frac{1}{32}(\cos[6\left(\frac{\pi}{2} - \theta\right)] + 6\cos[4\left(\frac{\pi}{2} - \theta\right)] + 15\cos[2\left(\frac{\pi}{2} - \theta\right)] + 10)$

$= \frac{1}{32}(\cos(\pi - 6\theta) + 6\cos(-4\theta) + 15\cos(\pi - 2\theta) + 10)$

$= \frac{1}{32}(-cos6\theta + 6cos4\theta - 15cos2\theta + 10)$

(3**) Express $-cos\theta$ in the form $cos\alpha$ (where α is to be found in terms of θ), using an algebraic method.

Solution

$-cos\theta = -\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\theta - \frac{\pi}{2}\right)$

$= \cos(\frac{\pi}{2} - \left[\theta - \frac{\pi}{2}\right]) = \cos(\pi - \theta)$ (or $\cos(3\pi - \theta)$ etc)

Alternatively, $-cos\theta = -\cos(-\theta) = -\sin\left(\frac{\pi}{2} - [-\theta]\right)$

$= \sin\left(-\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - \left[-\frac{\pi}{2} - \theta\right]\right) = \cos(\pi + \theta)$

(or $\cos(3\pi + \theta)$ etc)

(4**) Simplify $\sqrt{2(1 - cos\theta)}$ and $\sqrt{2(1 + cos\theta)}$

Solution

$\cos\theta = cos^2(\theta/2) - sin^2(\theta/2) = 1 - 2sin^2(\theta/2)$

so that $1 - \cos\theta = 2sin^2(\theta/2)$ and $\sqrt{2(1-cos\theta)} = 2\sin(\theta/2)$

Also, $\cos 2\theta = cos^2\theta - sin^2\theta = 2cos^2\theta - 1$, so that

$1 + \cos\theta = 2cos^2\left(\frac{\theta}{2}\right)$ and $\sqrt{2(1+cos\theta)} = 2\cos(\frac{\theta}{2})$

(5***) Show that

(i) $cos^4\theta - sin^4\theta = cos2\theta$

(ii) $cos^4\theta + sin^4\theta = 1 - \frac{1}{2}sin^2(2\theta)$

Solution

(i) $cos^4\theta - sin^4\theta = (cos^2\theta - sin^2\theta)(cos^2\theta + sin^2\theta)$

$= cos2\theta(1) = cos2\theta$

(ii) Consider

$1 = (cos^2\theta + sin^2\theta)^2 = cos^4\theta + sin^4\theta + 2cos^2\theta sin^2\theta$

Then $cos^4\theta + sin^4\theta = 1 - 2cos^2\theta sin^2\theta = 1 - \frac{1}{2}(2cos\theta sin\theta)^2$

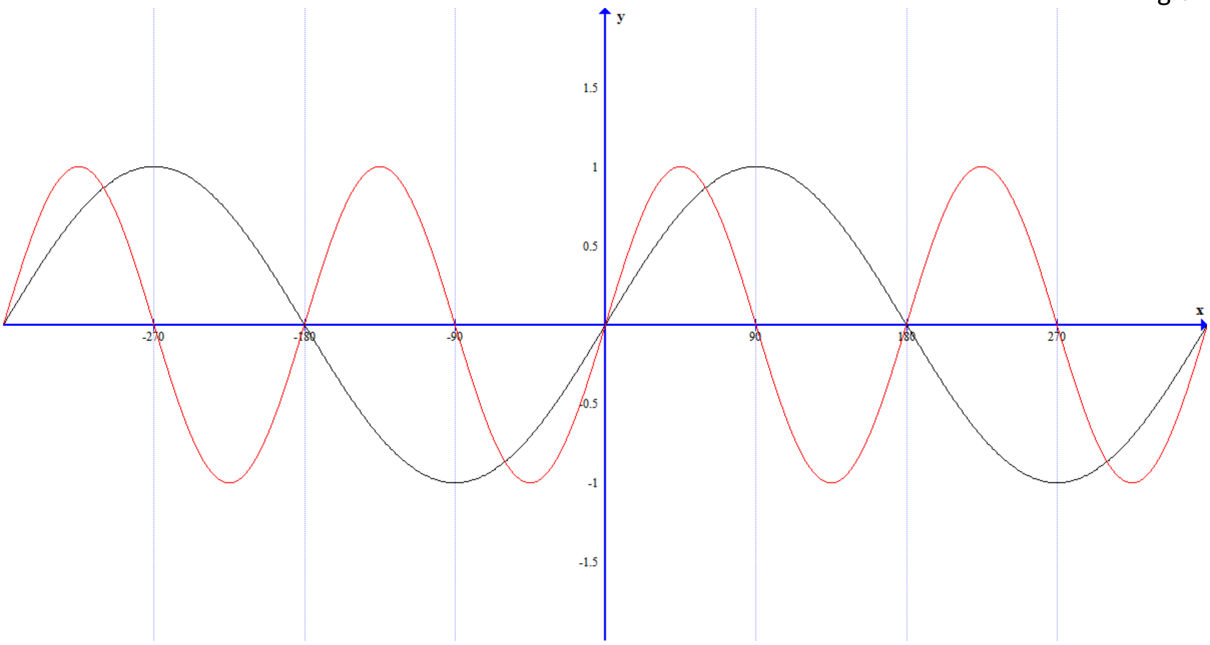
$= 1 - \frac{1}{2}sin^2(2\theta)$, as required.

(6**) Sketch $y = \sin(2x + 30°)$

Solution

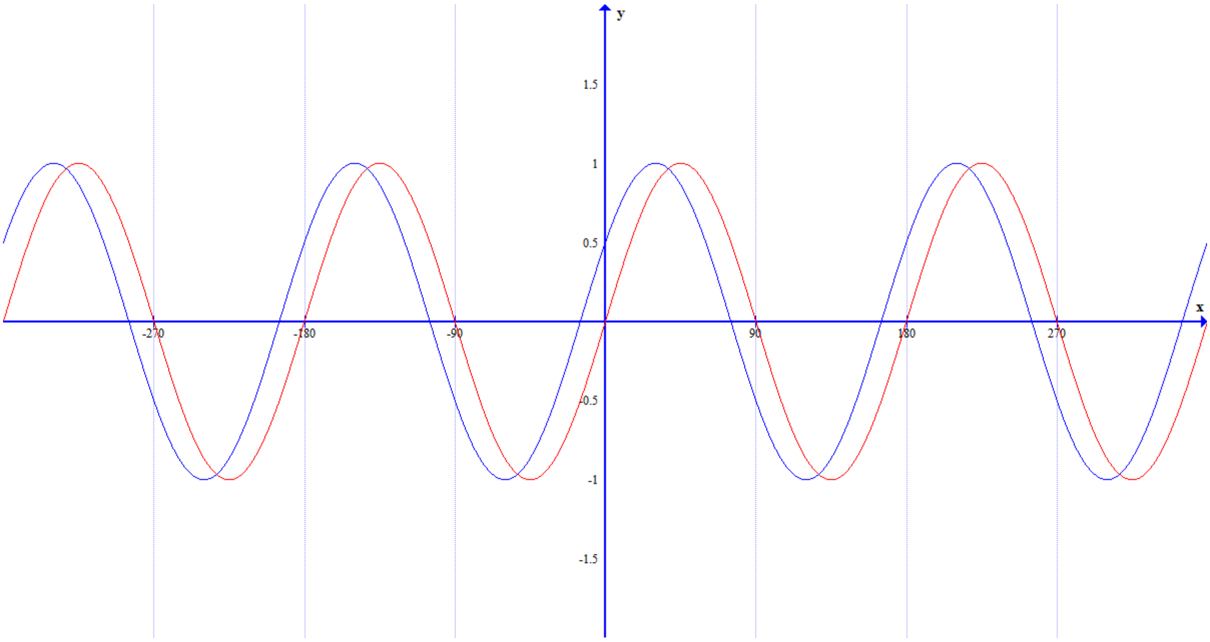
This is a composite transformation of $y = sinx$, and we have a choice of two approaches:

(i) $y = sinx \rightarrow y = sin2x$ [stretch of factor $\frac{1}{2}$ in the x-direction]

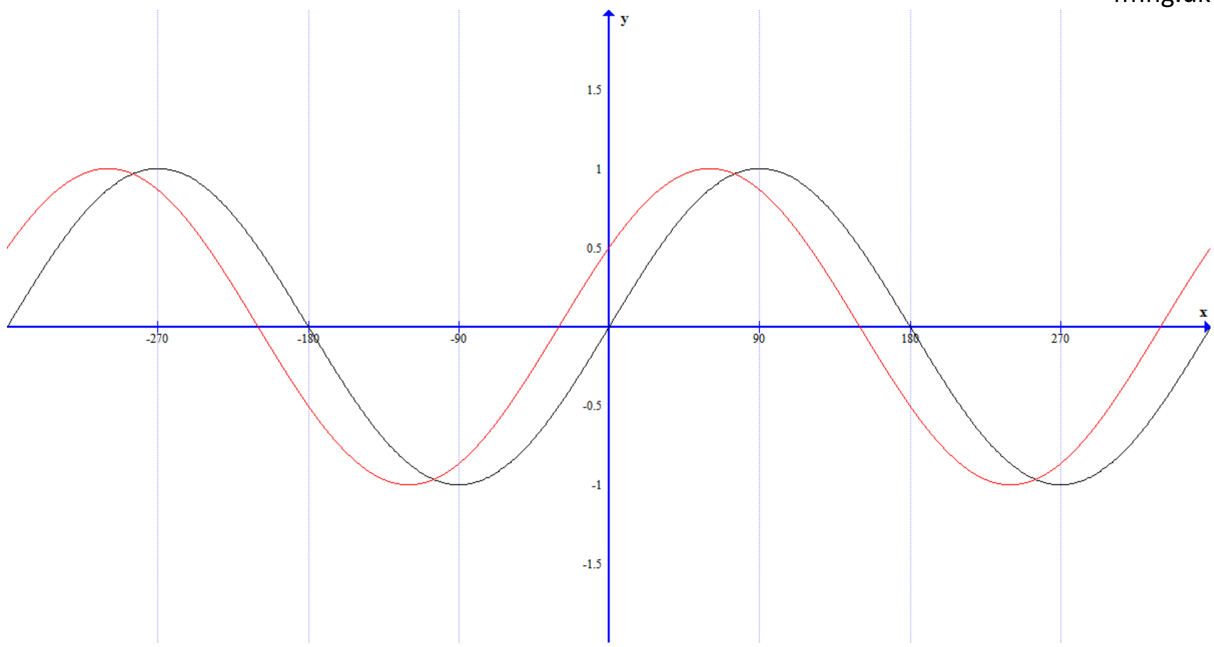


$\rightarrow y = \sin(2[x + 15°]$ [translation of 15° to the left]

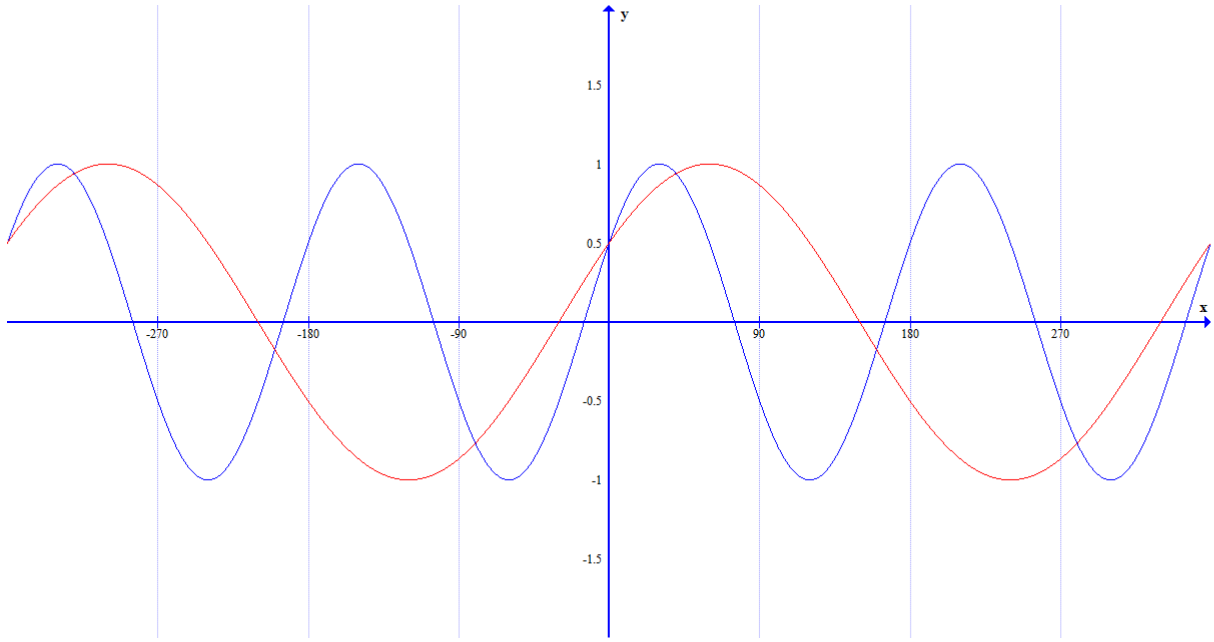
$= \sin(2x + 30°)$



(ii) $y = sinx \rightarrow y = \sin(x + 30°)$ [translation of 30° to the left]



$\rightarrow y = \sin(2x + 30)$ [stretch of factor $\frac{1}{2}$ in the x-direction]



Note that, in the above transformation, the graph 'pivots' about $x = 0$; ie $\sin(2x + 30°) = \sin(x + 30°)$ at $x = 0$.

You may find approach (i) easier to carry out.

(7*) If $sin\theta = 0.6$, where $0 \leq \theta < 360°$, find $tan\theta$

Solution

$tan\theta = \frac{sin\theta}{cos\theta} = \frac{sin\theta}{\pm\sqrt{1-sin^2\theta}} = \frac{\pm 0.6}{0.8} = \pm\frac{3}{4}$ (or draw graphs)

(8**) Show that each of (i)-(v) is true, by two methods:

(a) using the results (A)-(E) below
(b) applying translations and/or reflections to graphs

(i) $sin(\theta + 180) = -sin\theta$
(ii) $cos(180 - \theta) = cos(180 + \theta)$
(iii) $cos(90 - \theta) = -cos(90 + \theta)$
(iv) $sin(\theta - 180) = cos(\theta + 90)$
(v) $sin(\theta + 90) = cos\theta$

(A) $sin(-\theta) = -sin\theta$
(B) $sin(360 + \theta) = sin\theta$
(C) $sin(180 - \theta) = sin\theta$
(D) $sin\theta = cos(90 - \theta)$
(E) $cos(-\theta) = cos\theta$

Solution

(i)(a) $sin(\theta + 180) = sin(\theta + 180 - 360) = sin(\theta - 180)$

$= -sin(180 - \theta) = -sin\theta$

(b) Starting with the graph of $y = sin\theta$, $y = sin(\theta + 180)$ is obtained by a translation of 180° to the left, and this can be seen to be the graph of $y = -sin\theta$.

(ii)(a) $cos(180 - \theta) = \cos(\theta - 180) = \cos(\theta - 180 + 360)$

$= cos(180 + \theta)$

(b) Starting with the graph of $y = cos\theta$, $y = cos(\theta + 180)$ is obtained by a translation of 180° to the left, and this can be seen to have symmetry about the y-axis, so that replacing θ by $-\theta$ has no effect; ie $cos(\theta + 180) = cos(-\theta + 180) = cos(180 - \theta)$

(iii)(a) $cos(90 - \theta) = sin\theta = -\,sin(-\theta) = -cos(90 - [-\theta])$

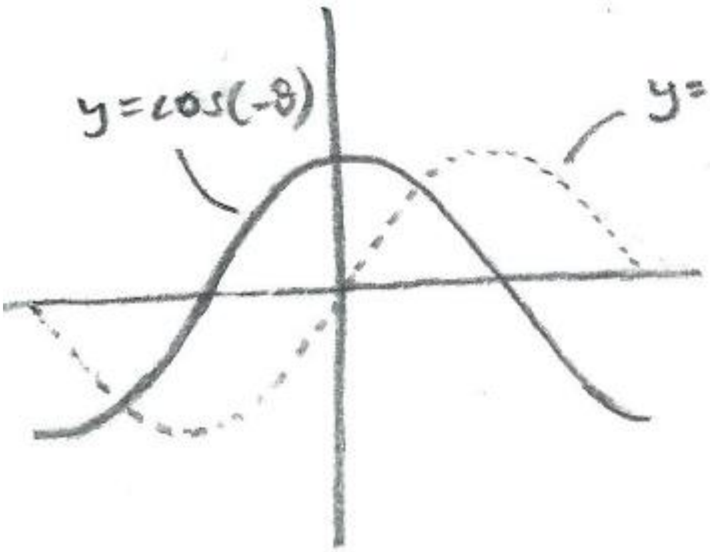
$= -cos(90 + \theta)$

(b) The graph of $y = cos(90 - \theta)$ can be obtained from $y = cos\theta$ by a reflection in the y-axis (having no effect), to give

$y = \cos(-\theta)$, followed by a translation of 90° to the right

(replacing θ with $\theta - 90$) , to give $y = \cos\big(-(\theta - 90)\big)$

$= cos(90 - \theta)$ (see diagram below).



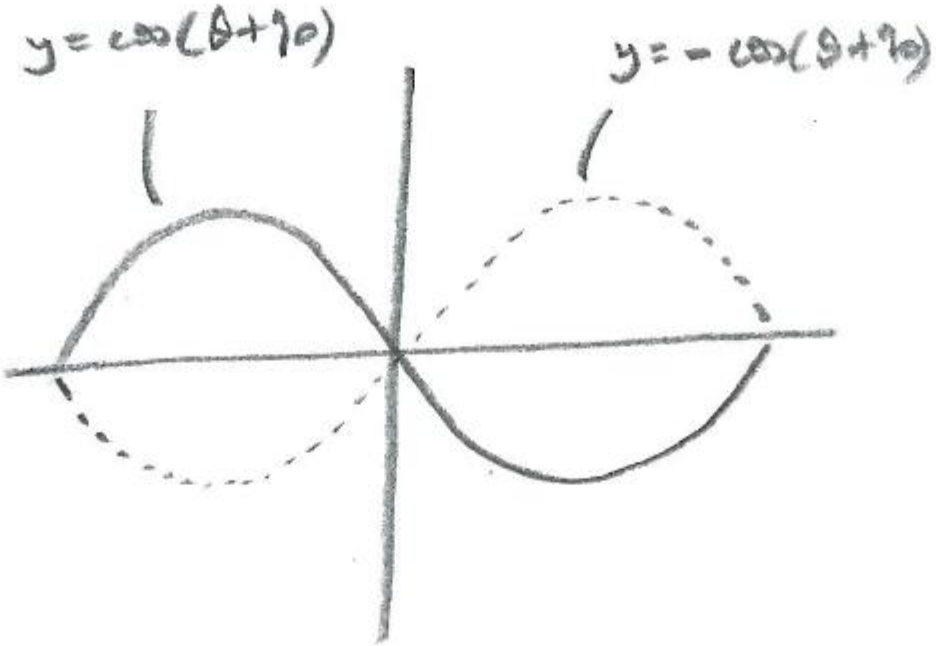
Then the graph of $y = -cos(90 + \theta)$ can be obtained from

$y = cos\theta$ by a translation of $90°$ to the left, to give

$y = cos(\theta + 90)$, followed by a reflection in the x-axis ,

to give $y = -cos(\theta + 90) = -cos(90 + \theta)$

(see diagram below).



And the graphs of $y = \cos\left(-(\theta - 90)\right)$ and $y = -cos(\theta + 90)$

are seen to be the same from these diagrams.

(iv)(a) $sin(\theta - 180) = sin(\theta - 180 + 360) = sin(\theta + 180)$

$= \cos(90 - [\theta + 180]) = \cos(-\theta - 90) = \cos(-[-\theta - 90])$

$= cos(\theta + 90)$

(b) The graph of $y = sin(\theta - 180)$ can be obtained from

$y = sin\theta$ by a translation of 180° to the right, whilst the graph of

$y = cos(\theta + 90)$ can be obtained from $y = cos\theta$ by a translation

of 90° to the left. The resulting graphs can be seen to be the same.

(v)(a) $sin(\theta + 90) = cos(90 - [\theta + 90]) = cos(-\theta) = cos\theta$

(b) The graph of $y = sin(\theta + 90)$ can be obtained from

$y = sin\theta$ by a translation of 90° to the left, which gives the graph

of $y = cos\theta$.