Transformations - Exercises (Sol'ns) (5 pages; 19/2/20)

(1***) Suppose that we wish to reflect $y = f(x)$ in the line $x = a$. What combination of transformations could be used to do this?

Solution

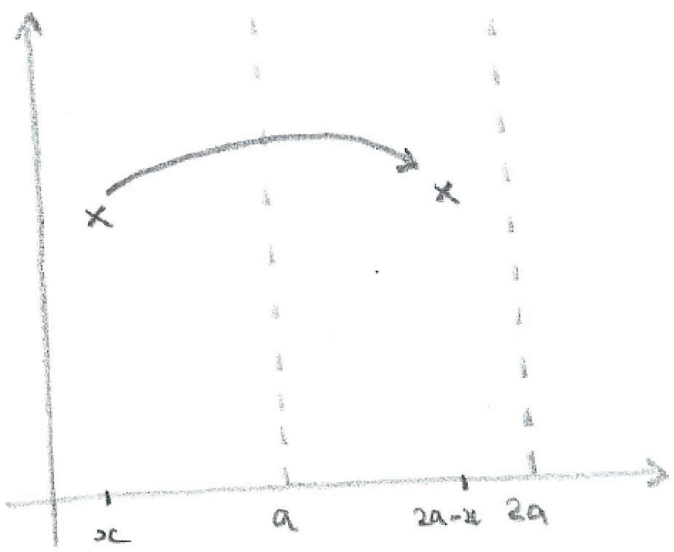
A particular point can be reflected in the line $x = a$ by considering a translation of a to the left, then performing a reflection in the y-axis and translating everything back, by a to the right.

In mathematical terms, x is first of all replaced by $x + a$; then x is replaced by $-x$, and finally x is replaced by $x - a$ (see note below).

Thus $f(x) \rightarrow f(x + a) \rightarrow f(-x + a) \rightarrow f(-[x - a] + a) = f(2a - x)$

[As an aid to memory, consider the reflection of $y = sinx$ about $x = \frac{\pi}{2}$, which is $y = \sin(\pi - x)$]

Alternative approach: $f(2a - x)$ can be justified by observing that when a point is reflected in the line $x = a$, its x coordinate changes from being x to the right of O (in the case where $x > 0$) to being x to the left of $2a$ (as in the example of $y = sinx$). See diagram below.



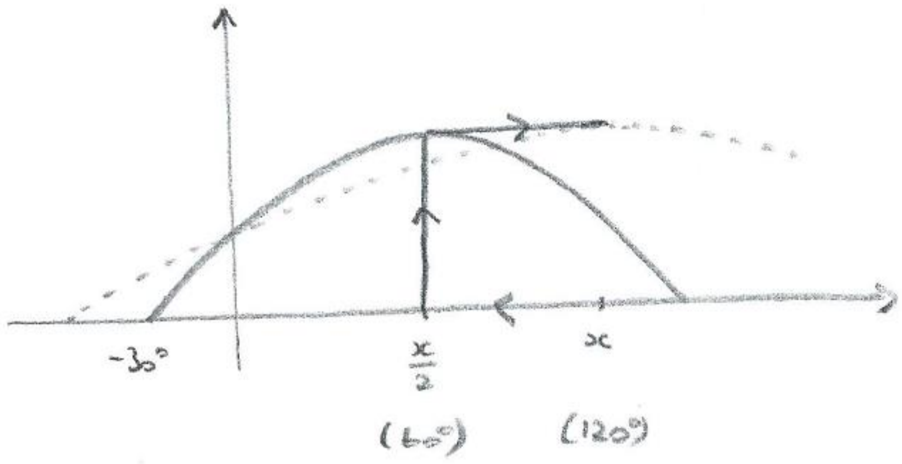
Note: An important point to observe when carrying out composite transformations is that, at any stage of the process, only the following operations are allowed: replacing x with $x + a$ (where a can be negative), or replacing x with kx (where k can be negative).

For example, if we want to stretch $y = \sin(x + 30°)$ by a scale factor 2 in the x-direction, then the point $(x, \sin(x + 30°)\ )$ is moving to $(2x, \sin(x + 30°))$. Making the substitution $u = 2x$, the coordinates of this point on the new curve are

$(u, \sin\left(\frac{u}{2} + 30°\right))$, and re-labelling, to give y as a function of x (rather than u), we have $y = \sin(\frac{x}{2} + 30°)$.

Alternatively (going the other way): the graph of

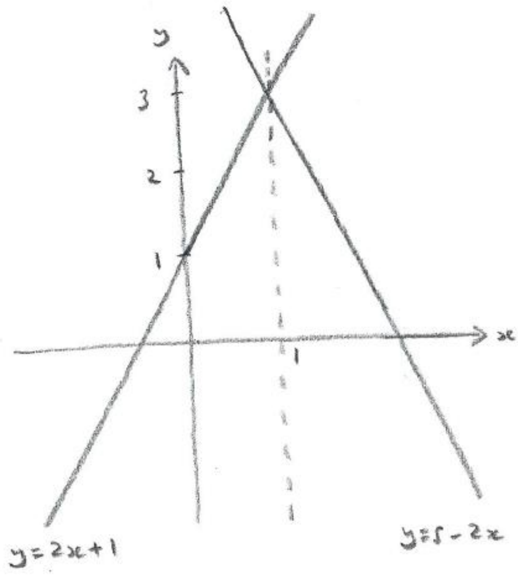
$y = \sin(\frac{x}{2} + 30°)$ can be obtained as follows: we want the curve with coordinates $(x, \sin(\frac{x}{2} + 30°)\ )$. This can be obtained from the curve with coordinates $(x, \sin(x + 30°)\ )$ by 'looking to the left' of x, to find the point $\left(\frac{x}{2}, \sin\left(\frac{x}{2} + 30°\right)\right)$, and then dragging it back to the right, to give $\left(x, \sin\left(\frac{x}{2} + 30°\right)\right)$ [see diagram below]. (Note that, as this transformation is a stretch, the amount of dragging will depend on the distance from the Origin.) The dragging to the right explains why we see the curve stretching outwards (even though x is being replaced by $\frac{x}{2}$). A similar argument applies in the case of translations (though here the amount of dragging is the same for all points).



(2***) Find the equation of the line resulting from the reflection of $y = 2x + 1$ in the line $x = 1$.

Solution

The transformed line is $y = 2(2 - x) + 1 = 5 - 2x$



Check: The transformed line will pass through the point where $y = 2x + 1$ meets the line $x = 1$; ie at $(1, 3)$, and will have a gradient of -2; hence its equation is $\frac{y-3}{x-1} = -2$ etc

(3**) Describe the transformation represented by $y = e^x \rightarrow$ $y = e^{4-x}$

Solution

Step 1: Replace x with $-x$ (reflection in y-axis), to give $y = e^{-x}$

Step 2: Replace x with $x - 4$ (translation of $\binom{4}{0}$), to give

$y = e^{-(x-4)} = e^{4-x}$

So the transformation is a reflection in the y-axis, followed by a translation of $\binom{4}{0}$. This enables the graph to be sketched.
However, this compound transformation can be represented as a single transformation: in general, a reflection in the line $x = L$ is achieved by replacing x with $2L - x$, so that in this case we have a reflection in the line $x = 2$. [Consider the statement $\sin(\pi - \theta) = sin\theta$, which arises because of the symmetry of the sine curve about $\theta = \frac{\pi}{2}$.]

(4**) What happens to the graph of $y = f(x)$ when it is transformed to:

(a) $y = f(|x|)$ (b) $|y| = f(x)$

Solution

(a) When $x \geq 0$, $f(|x|) = f(x)$; when $x < 0$, $f(|x|) = f(-x)$; ie that part of $y = f(x)$ to the right of the y-axis is reflected in the y-axis.

So $y = f(|x|)$ is the right half of $y = f(x)$, together with its reflection in the y-axis.

(b) First of all, $|y| = f(x)$ is only defined for x such that $f(x) \geq 0$.

The graph of $|y| = f(x)$ is similar to that of $y^2 = f(x)$, or

$y = \pm\sqrt{f(x)}$, in that it has two branches: $y = f(x)$ and

$y = -f(x)$.

So, provided $f(x) \geq 0$, $|y| = f(x)$ is the same as $y = f(x)$, with the addition of its reflection in the x-axis.

(5**) What combination of transformations converts $y = 2^x$ to

$y = 2^{4x-2}$?

Solution

$y = 2^x \rightarrow y = 2^{4x}$ is a stretch of scale factor $\frac{1}{4}$ in the x-direction

Then $y = 2^{4x} \rightarrow y = 2^{4(x-\frac{1}{2})} = 2^{4x-2}$ is a translation of $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$

[Alternatively, $y = 2^{4x} \rightarrow y = \left(\frac{1}{4}\right) 2^{4x} = 2^{4x-2}$ is a stretch of scale factor $\frac{1}{4}$ in the y-direction.]

(6*) Find the equation of the function resulting from a translation of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ of $y = 2x + 1$

Solution

$y = 2x + 1 \rightarrow y = [2(x - 1) + 1] + 2 = 2x + 1$