# **Transformations of Functions** (9/9/2013)

## (A) Translation of the form f(x) → f(x+k) or f(x-k)

**Example:** How will the graph of $y = (x-2)^2$ be related to $y = x^2$?

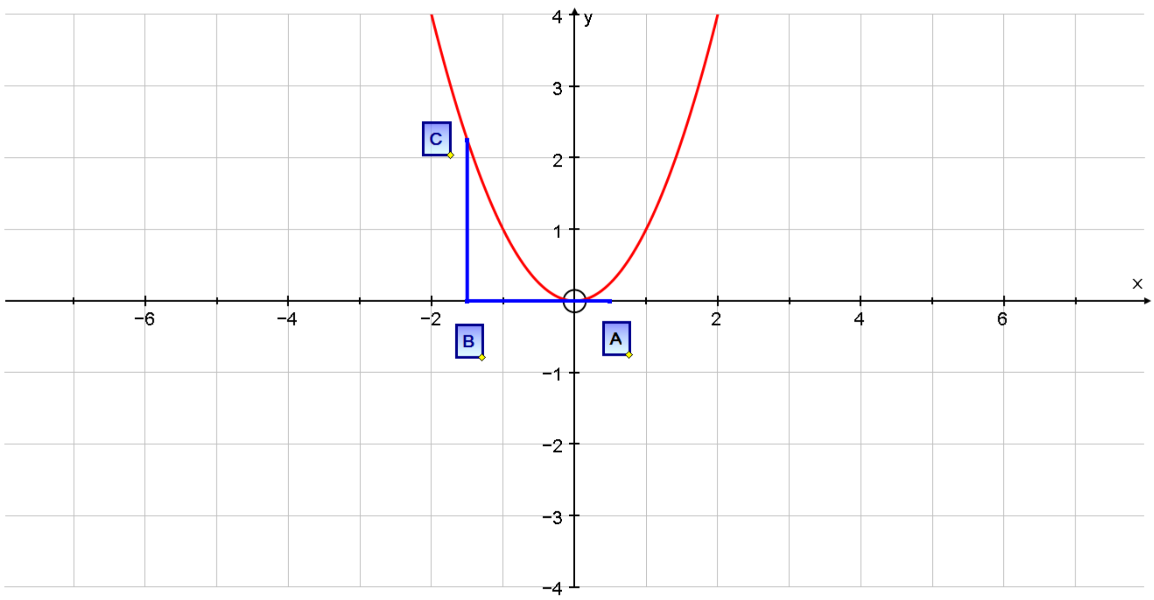
We are trying to find the graph with general coordinates $(x, (x-2)^2)$.

This can be done in stages, as follows:

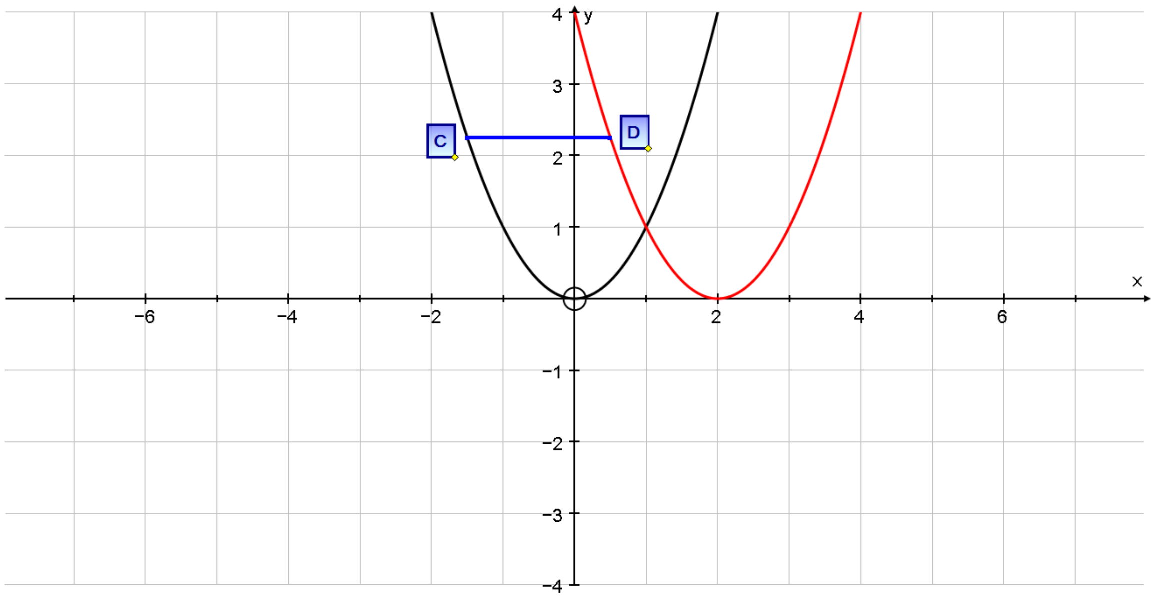
Start with the point (x, 0). [In the diagram below, x is taken to be 0.5]

Then look to the left to find the point (x-2, 0).

The graph of $y = x^2$ then gives us the point $(x-2, (x-2)^2)$.




Then dragging this point to the right gives us $(x, (x-2)^2)$, as required. The same process can be applied to any point, to give the new position of the graph, as shown below:




Note that it is the 'dragging over to the right' that causes the resulting curve to be the opposite of what might be expected (the subtraction sign in 'x-2' might suggest a movement to the left).

In general, this transformation is described as a "translation of k to the right"

and can be represented by the vector $\binom{k}{0}$

(obviously if k is positive there will be a translation to the left).

**(B) Stretch of the form f(x) → f(kx)**

[k could be greater than or less than 1, and/or negative]

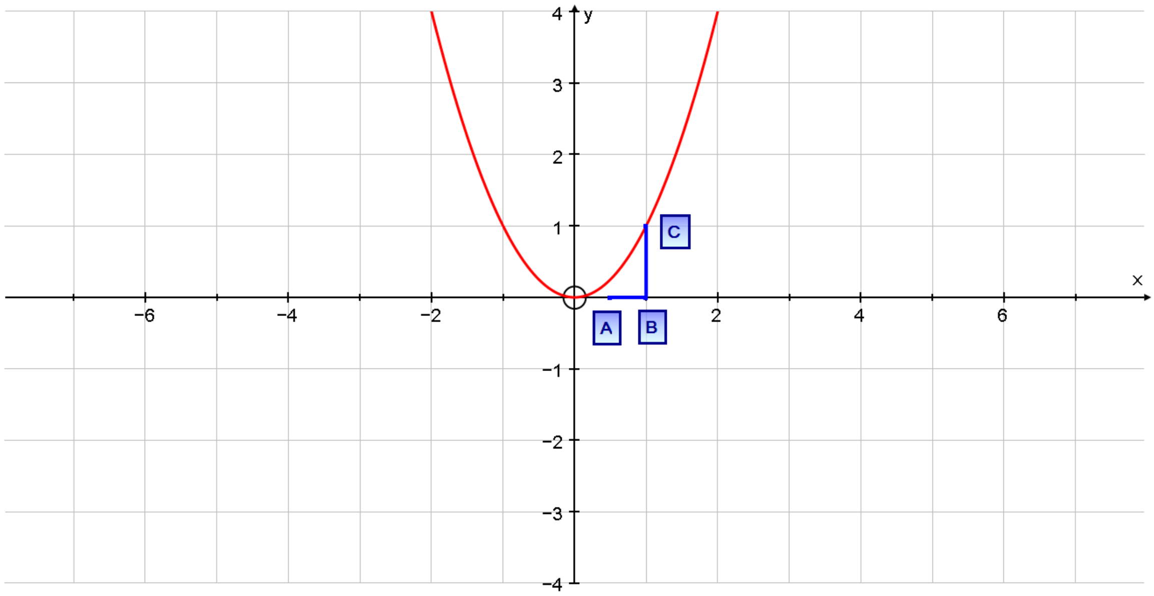
A similar argument applies as for the translation f(x) → f(x+k).

Example: $y = x^2 \rightarrow y = (2x)^2$

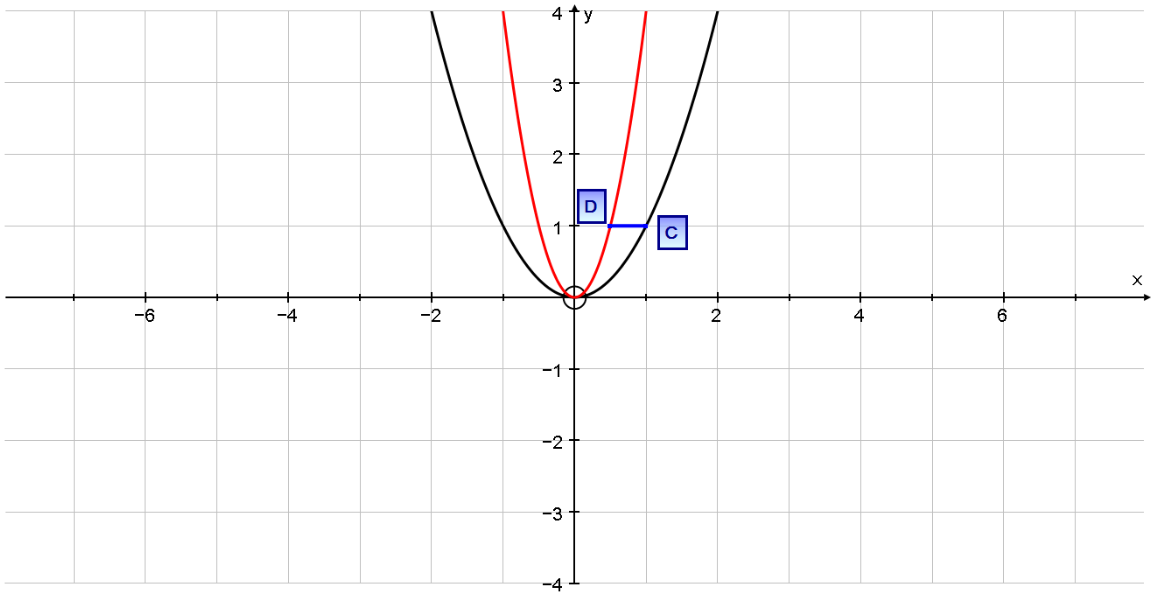
Start with the point (x, 0).

Then look to the right to find the point (x, 2x).

The graph of $y = x^2$ then gives us the point $(2x, (2x)^2)$.




Then dragging this point to the left gives us $(x, (2x)^2)$, as shown below:




There are, however, two important differences when compared with the translation case.

First of all, when x = 0 no movement occurs (because 2x = x), and the amount of movement increases with the size of x; in other words, the graph is compressed, rather than translated.

Secondly, for negative values of x the movement is to the right (because multiplying by 2 involves 'looking to the left' [at a larger negative number] and dragging to the right). We can think of this as dragging the graph towards x = 0 for both positive and negative x.

This type of transformation is described as a stretch of scale factor 1/2 (for this example), parallel to the x axis (or "in the direction of the x axis"). Note that the stretch factor refers to what you actually see happen to the graph.

In the case of the transformation $f(x) \to f(1/2.x)$, we have a stretch of scale factor 2 parallel to the x axis, and the graph is seen to expand. Avoid describing this transformation as an enlargement, as this is reserved for cases where a stretch of the same size occurs in both the x and y directions.

Where k is negative, there is a reflection in the y axis, in addition to any stretching or compressing.

**(C) Stretch of the form $f(x) \to kf(x)$**

Note that the function stays the same at y = 0, and that the graph is stretched away from the x axis (ie the negative values of y become more negative).

The transformation is described as a "stretch of scale factor k parallel to the y axis" (or "in the direction of the y axis").

As with the other transformations, the description refers to what you actually see happen, but in this case it is what you would expect: $f(x) \to 2f(x)$ means that the graph stretches outwards.

Note that the transformation $y = x^2 \to y = 2x^2$ has a similar effect to that of

$y = x^2 \to y = (2x)^2 = 4x^2$, despite the fact that the first one is a transformation "parallel to the y axis", whilst the second is a transformation "parallel to the x axis".

## (D) Compound transformations

**Example 1**: To obtain the curve $y = ax^2 + bx + c$ from $y = x^2$

(i) Complete the square: $y = a(x + \frac{b}{2a})^2 + d$ $[d = c - \frac{b^2}{4a}]$

(ii) Translate $y = x^2$ to $y = (x + \frac{b}{2a})^2$ by translating the curve $\frac{b}{2a}$ to the left

(iii) Obtain $y = a(x + \frac{b}{2a})^2$ by stretching by a factor of (a) parallel to the y axis

(iv) Translate upwards by d

**Example 2**: To obtain the curve y = sin (2x + 30) from y = sin x, note that the only permissible steps are those that involve replacing x with x+k or with kx.

Thus the following are allowed:

(a) Translation of 30° to the left [ sin x → sin (x+30) ]

followed by a stretch of scale factor ½ parallel to the x axis

[ → sin (2x + 30)]

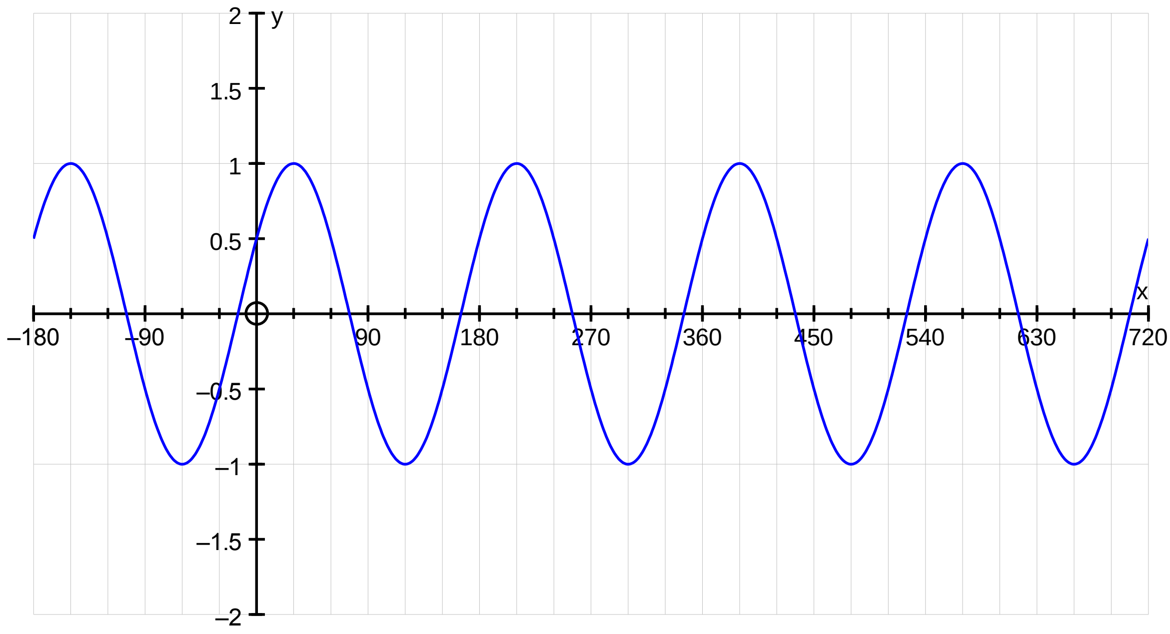
Note that the 30 is not multiplied by 2; the only thing that is happening is that x is being replaced by 2x.

(b) A stretch of scale factor ½ parallel to the x axis [ sin x → sin (2x) ]

followed by a translation of 15° to the left [→ sin 2(x+15) = sin (2x + 30)]

[x is being replaced by x+15]

It is probably easier to visualise (b) than (a).




y = sin(2x+30)

As a check, it may help to look at what happens to certain critical points; so in the above example:

x = -15 ⇒ sin(2x+30) = sin(0) = 0

x = 30 ⇒ sin(2x+30) = sin(90) = 1

x = 75 ⇒ sin(2x+30) = sin(180) = 0

x = 120 ⇒ sin(2x+30) = sin(270) = -1

x = 165 ⇒ sin(2x+30) = sin(360) = 0

Note that these critical points were chosen to make 2x+30 a convenient value.