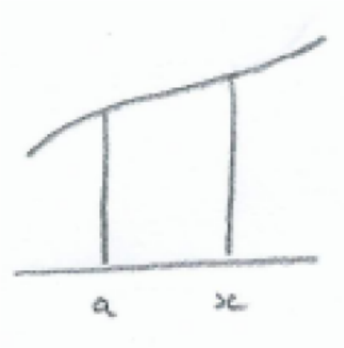
Taylor Series (7 pages; 1/6/26)

(1) Maclaurin series are a special case of Taylor series, of which there are two versions.

The following is intended as a simple way of finding the first couple of terms of the Taylor series, in each case. The remaining terms then follow, once the pattern has been established.

Centred on $x = a$ (version 1)



For x close to a, $f'(a) \approx \frac{f(x)-f(a)}{x-a}$,

leading to $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} + \cdots$

Centred on $x = a$ (version 2)

For x close to a, $f'(a) \approx \frac{f(a+x)-f(a)}{x}$

leading to $f(x+a) = f(a) + xf'(a) + \frac{x^2 f''(a)}{2!} + \cdots$

Centred on $x = 0$ (Maclaurin expansion)

For x close to 0, $f'(0) \approx \frac{f(x)-f(0)}{x}$

leading to $f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \cdots$

(This is also obtained by putting $a = 0$ in versions 1 or 2.)

(2) More formal derivation of the Taylor series expansion:

Suppose that $g(x) = g(0) + g'(0)x + g''(0)\frac{x^2}{2!} + \cdots$ (A)

Define $f(x+a) = g(x)$

[eg $g(x) = \ln(1+x) = f(x+1)$, where $a = 1$]

Then $g'(x) = \frac{d}{dx}f(x+a) = \frac{d}{d(x+a)}f(x+a).\frac{d}{dx}(x+a)$

$= f'(x+a)$

[Note that $f'(x+a)$ means the derivative wrt $x+a$; not wrt x]

Also $g''(x) = \frac{d}{dx}g'(x) = \frac{d}{dx}f'(x+a)$

$= \frac{d}{d(x+a)}f'(x+a).\frac{d}{dx}(x+a) = f''(x+a)$, and so on.

Thus, from (A),

$f(x+a) = f(0+a) + f'(0+a)x + f''(0+a)\frac{x^2}{2!} + \cdots$

$= f(a) + f'(a)x + f''(a)\frac{x^2}{2!} + \cdots$

(version 2 of the Taylor Series above)

Also, if we write $X = x + a$, this becomes

$f(X) = f(a) + f'(a)(X-a) + f''(a)\frac{(X-a)^2}{2!} + \cdots$

(version 1 of the Taylor Series above)

(3) To establish the Taylor series for $f(x) = lnx$ about $x = 1$:

Version 1:

$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \cdots$

$f(1) = 0$

$f'(x) = \frac{1}{x}$; $f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$; $f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$; $f''(1) = 2$

Thus $lnx = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$

[Note: This can also be obtained from the Maclaurin series, by writing $lnx = \ln{(1 + [x-1])}$]

Version 2: $\ln{(x+1)} = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

[In order to create a Taylor series for $f(x)$ about $x = a$, we have to make one of two compromises: either express $f(x)$ in terms of powers of $x - a$, or obtain a series for $f(x + a)$.]

(4) To establish the Taylor series for $f(x) = \frac{1}{x}$ about $x = 1$:

$f'(x) = -\frac{1}{x^2}$; $f''(x) = \frac{2}{x^3}$; $f'''(x) = -\frac{3!}{x^4}$

Version 1:

$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \cdots$

ie $\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$

Version 2:

$\frac{1}{x+1} = 1 - x + x^2 - x^3 + \cdots$

(5) Taylor expansions of solutions to differential equations can be found as follows.

Example: $\frac{dy}{dx} + y - e^x = 0$, where $y = 2$ when $x = 0$

$\frac{dy}{dx} = e^x - y$

$\frac{dy}{dx}\ |(x = 0)\ \ =\ 1 - 2 = -1$

$\frac{d^2y}{dx^2} = e^x - \frac{dy}{dx}$

$\frac{d^2y}{dx^2}|(x = 0) = 1 - (-1) = 2$

$\frac{d^3y}{dx^3} = e^x - \frac{d^2y}{dx^2}$

$\frac{d^3y}{dx^3}|(x = 0) = 1 - 2 = -1$

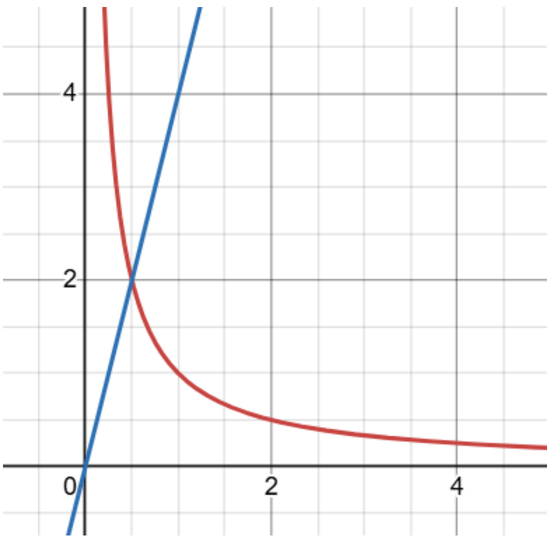
Then $y = y_0 + (x - x_0)\frac{dy}{dx}\ |(x = 0) + \frac{\frac{d^2y}{dx^2}|(x=0)\ (x-x_0)^2}{2!}$

$+ \frac{\frac{d^3y}{dx^3}|(x=0)\ (x-x_0)^3}{3!} + \cdots$

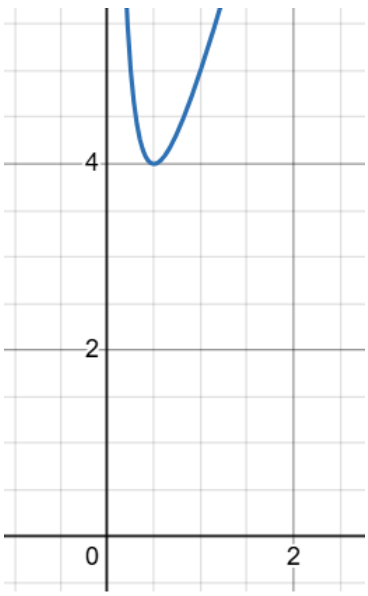
$= 2 - x + x^2 - \frac{x^3}{6} + \cdots$

[This can be extended to higher order equations, when $\frac{dy}{dx}\ |(x = 0)$ etc are given.]

(6) Example: $f(x) = 4x + \frac{1}{x}$; $x > 0$



$y = 4x\ (blue)\ \&\ y = \frac{1}{x}\ (red)$



$y = 4x + \frac{1}{x}$

From the graphs of $y = 4x$ & $y = \frac{1}{x}$, we see that $f(x) \to \infty$ as

$x \to 0^+$ (ie tending to zero from above) and $f(x) \to \infty$ as $x \to \infty$. And so, as $f(x)$ is continuous for $x > 0$, there must be a minimum somewhere. The only point of any significance is where $y = 4x$ and $y = \frac{1}{x}$ intersect (when $x = \frac{1}{2}$), and this is suspected to be the minimum.

The gradients of $y = 4x$ and $y = \frac{1}{x}$ are 4 and $-\frac{1}{x^2}$ respectively.

At $x = \frac{1}{2} + \delta$ (where δ is a very small positive number), these gradients are 4 and $-\frac{1}{\left(\frac{1}{2}+\delta\right)^2} > -4$, so that $f(x)$ increases as x increases from $\frac{1}{2}$ to $\frac{1}{2} + \delta$ (the increase in $y = 4x$ outweighs the reduction in $y = \frac{1}{x}$).

And at $x = \frac{1}{2} - \delta$, the gradients are 4 and $-\frac{1}{\left(\frac{1}{2}-\delta\right)^2} < -4$, so that $f(x)$ decreases as x increases from $\frac{1}{2} - \delta$ to $\frac{1}{2}$ (the increase in $y = 4x$ is outweighed by the reduction in $y = \frac{1}{x}$). And so, as x **decreases** from $\frac{1}{2}$ to $\frac{1}{2} - \delta$, $f(x)$ increases.

Thus there is a (local) minimum at $x = \frac{1}{2}$.

From version 1 of the Taylor series for $f(x)$,

$$f(x) = f\left(\frac{1}{2}\right) + \left(x - \frac{1}{2}\right) f'\left(\frac{1}{2}\right) + \frac{\left(x-\frac{1}{2}\right)^2 f''\left(\frac{1}{2}\right)}{2!} + \cdots$$

Now $f'(x) = 4 - \frac{1}{x^2}$; $f''(x) = \frac{2}{x^3}$, and $f'''(x) = -\frac{3!}{x^4}$,

so that $f'(\frac{1}{2}) = 0$; $f''(x) = 2(8)$, and $f'''(x) = -3!\,(16)$, and $f(x) = 4 + 8\left(x - \frac{1}{2}\right)^2 - 16\left(x - \frac{1}{2}\right)^3 + \cdots$

For x sufficiently close to $\frac{1}{2}$, the $8\left(x - \frac{1}{2}\right)^2$ term will dominate subsequent terms (bearing in mind that, for $|u| < 1, u^2 > u^3$),

so that $f(x) > f(\frac{1}{2})$ for x sufficiently close to $\frac{1}{2}$.

This is a particular case of the following general result:

A necessary and sufficient condition for a turning point is that the first non-zero derivative of the function must be of even order ($\geq$ 2). The sign of this derivative then determines whether it is a maximum (if negative) or minimum (if positive).

Here $f'\left(\frac{1}{2}\right) = 0$ and $f''\left(\frac{1}{2}\right) = 16$.