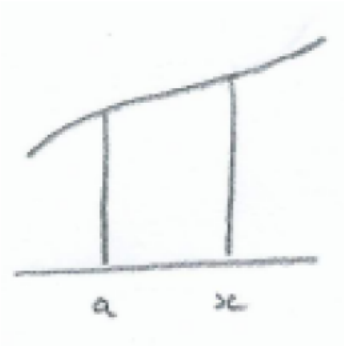
Taylor Series (5 pages; 23/7/21)

(1) Maclaurin series are a special case of Taylor series, of which there are two versions.

The following is intended as a simple way of finding the first couple of terms of the Taylor series, in each case. The remaining terms then follow, once the pattern has been established.

Centred on $x = a$ (version 1)



For x close to a, $f'(a) \approx \frac{f(x)-f(a)}{x-a}$,

leading to $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} + \cdots$

Centred on $x = a$ (version 2)

For x close to a, $f'(a) \approx \frac{f(a+x)-f(a)}{x}$

leading to $f(x+a) = f(a) + xf'(a) + \frac{x^2 f''(a)}{2!} + \cdots$

Centred on $x = 0$ (Maclaurin expansion)

For x close to 0, $f'(0) \approx \frac{f(x)-f(0)}{x}$

leading to $f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \cdots$

(This is also obtained by putting $a = 0$ in versions 1 or 2.)

(2) More formal derivation of the Taylor series expansion:

Suppose that $g(x) = g(0) + g'(0)x + g''(0)\frac{x^2}{2!} + \cdots$ (A)

Define $f(x+a) = g(x)$

[eg $g(x) = \ln(1+x) = f(x+1)$, where $a = 1$]

Then $g'(x) = \frac{d}{dx}f(x+a) = \frac{d}{d(x+a)}f(x+a).\frac{d}{dx}(x+a)$

$= f'(x+a)$

[Note that $f'(x+a)$ means the derivative wrt $x+a$; not wrt x]

Also $g''(x) = \frac{d}{dx}g'(x) = \frac{d}{dx}f'(x+a)$

$= \frac{d}{d(x+a)}f'(x+a).\frac{d}{dx}(x+a) = f''(x+a)$, and so on.

Thus, from (A),

$f(x+a) = f(0+a) + f'(0+a)x + f''(0+a)\frac{x^2}{2!} + \cdots$

$= f(a) + f'(a)x + f''(a)\frac{x^2}{2!} + \cdots$

(version 2 of the Taylor Series above)

Also, if we write $X = x + a$, this becomes

$f(X) = f(a) + f'(a)(X-a) + f''(a)\frac{(X-a)^2}{2!} + \cdots$

(version 1 of the Taylor Series above)

(3) To establish the Taylor series for $f(x) = lnx$ about $x = 1$:

Version 1:

$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \cdots$

$f(1) = 0$

$f'(x) = \frac{1}{x}\ ; f'(1) = 1$

$f''(x) = -\frac{1}{x^2}; f''(1) = -1$

$f'''(x) = \frac{2}{x^3}; f''(1) = 2$

Thus $lnx = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$

[Note: This can also be obtained from the Maclaurin series, by writing $lnx = \ln(1 + [x-1])$]

Version 2: $\ln{(x+1)} = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

[In order to create a Taylor series for $f(x)$ about $x = a$, we have to make one of two compromises: either express $f(x)$ in terms of powers of $x - a$, or obtain a series for $f(x+a)$.]

(4) To establish the Taylor series for $f(x) = \frac{1}{x}$ about $x = 1$:

$f'(x) = -\frac{1}{x^2}$; $f''(x) = \frac{2}{x^3}$; $f'''(x) = -\frac{3!}{x^4}$

Version 1:

$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \cdots$

ie $\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$

Version 2:

$\frac{1}{x+1} = 1 - x + x^2 - x^3 + \cdots$

(5) Taylor expansions of solutions to differential equations can be found as follows.

Example: $\frac{dy}{dx} + y - e^x = 0$, where $y = 2$ when $x = 0$

$\frac{dy}{dx} = e^x - y$

$\frac{dy}{dx}\ |(x=0) \ = \ 1 - 2 = -1$

$\frac{d^2y}{dx^2} = e^x - \frac{dy}{dx}$

$\frac{d^2y}{dx^2}\,|(x=0) = 1 - (-1) = 2$

$\frac{d^3y}{dx^3} = e^x - \frac{d^2y}{dx^2}$

$\frac{d^3y}{dx^3}\,|(x=0) = 1 - 2 = -1$

Then $y = y_0 + (x - x_0)\frac{dy}{dx}\ |(x=0) + \frac{\frac{d^2y}{dx^2}|(x=0)\ (x-x_0)^2}{2!}$

$+ \frac{\frac{d^3y}{dx^3}|(x=0)\ (x-x_0)^3}{3!} + \cdots$

$= 2 - x + x^2 - \frac{x^3}{6} + \cdots$

[This can be extended to higher order equations, when $\frac{dy}{dx}\ |(x=0)$ etc are given.]