TMUA: Recommended Past TMUA & MAT Questions

(50 Pages; 10/1/25)

Notes:

(i) MAT multiple choice questions are excellent practice for the harder TMUA ones.

(ii) See also TMUA Methods: Ideas & Exercises.

Contents

MAT 2011, Q1/A [Cubics]

TMUA 2021, P2, Q20 [Integration]

TMUA 2016, P2, Q4 [Logic]

TMUA 2016, P2, Q10 [Proof]

TMUA Specimen P2, Q18 [Statistics]

TMUA Specimen P2, Q19 [Polynomials]

19. The positive real numbers a, b, and c are such that the equation

$$x^3 + ax^2 = bx + c$$

has three real roots, one positive and two negative.

Which one of the following correctly describes the real roots of the equation

$$x^3 + c = ax^2 + bx\ ?$$

A It has three real roots, one positive and two negative.

B It has three real roots, two positive and one negative.

C It has three real roots, but their signs differ depending on a, b, and c.

D It has exactly one real root, which is positive.

E It has exactly one real root, which is negative.

F It has exactly one real root, whose sign differs depending on a, b, and c.

G The number of real roots can be one or three, but the number of roots differs depending on a, b, and c.

Solution

[A graphical approach looks promising here, but unfortunately turns out to be too complicated.]

The two equations can be rewritten as:

$x^3 + ax^2 - bx - c = 0$ (1)

and $x^3 - ax^2 - bx + c = 0$ (2)

[Given that only the signs of even powers of x differ]

Let $y = -x$

Then (2) becomes $-y^3 - ay^2 + by + c = 0$

or $y^3 + ay^2 - by - c = 0$, which has the same roots as (1).

So, if (1) has one positive and two negative roots, (2) will have one negative and two positive roots.

Answer: B

Comments

Example of rearrangement (substitution)[after observing that only the signs of even powers of x differ]

TMUA, Specimen P1, Q10 [Transformations]

10. The curve $y = \cos x$ is reflected in the line $y = 1$ and the resulting curve is then translated by $\frac{\pi}{4}$ units in the positive x-direction. The equation of this new curve is

A $y = 2 + \cos\left(x + \frac{\pi}{4}\right)$

B $y = 2 + \cos\left(x - \frac{\pi}{4}\right)$

C $y = 2 - \cos\left(x + \frac{\pi}{4}\right)$

D $y = 2 - \cos\left(x - \frac{\pi}{4}\right)$

Solution

Reflecting $y = f(x)$ in the line $y = b$ can be shown to give $2b - y = f(x)$ [reflecting $y = f(x)$ in the line $x = a$ gives

$y = f(2a - x)$]

Proof: The reflection in $y = b$ is equivalent to a translation of $\begin{pmatrix} 0 \\ -b \end{pmatrix}$, followed by a reflection in the x-axis, and then a translation of $\begin{pmatrix} 0 \\ b \end{pmatrix}$: this produces $y = f(x) \rightarrow y = f(x) - b \rightarrow -(f(x) - b)$

$\rightarrow -(f(x) - b) + b = 2b - f(x)$

When $f(x) = cosx$ and $b = 1$ this gives $y = 2 - cosx$

Then a translation of $\begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$ gives $y = 2 - \cos\left(x - \frac{\pi}{4}\right)$,

Answer: D

TMUA, Specimen P1, Q13 [Polynomials]

13. How many real roots does the equation $x^4 - 4x^3 + 4x^2 - 10 = 0$ have?

A 0

B 1

C 2

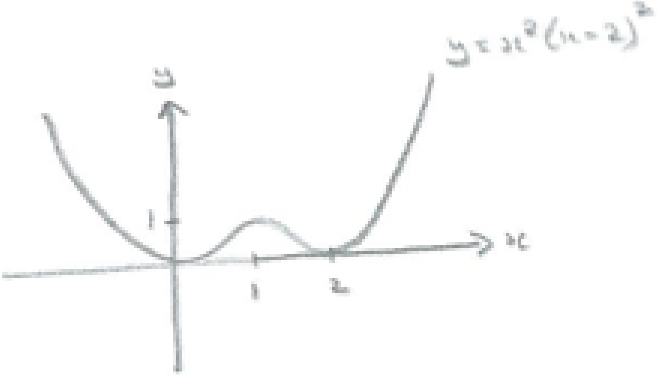
D 3

E 4

Solution

Equivalently, consider the roots of $x^2(x^2 - 4x + 4) = 10$

ie $x^2(x-2)^2 = 10$



Referring to the graph, there are 2 roots.

[$y = f(x) = x^2(x-2)^2$ has symmetry about $x = 1$, as the translation of $f(x)$ by 1 to the left is

$g(x) = f(x+1) = (x+1)^2(x-1)^2$,

and $g(-x) = (-x+1)^2(-x-1)^2 = (x-1)^2(x+1)^2 = g(x)$,

and thus $g(x)$ is an even function (with symmetry about the y-axis)]

Answer: C

TMUA 2020, P1, Q13 [Polynomials]

13 How many real roots does the equation $3x^5 - 10x^3 - 120x + 30 = 0$ have?

A 1

B 2

C 3

D 4

E 5

Solution

Writing $f(x) = 3x^5 - 10x^3 - 120x + 30$,

$f'(x) = 15x^4 - 30x^2 - 120$

Then $f'(x) = 0 \Rightarrow (x^2 - 4)(x^2 + 2) = 0 \Rightarrow x = \pm 2$

$f''(x) = 60x^3 - 60x$

$f''(-2) < 0 \Rightarrow$ maximum

$f''(2) > 0 \Rightarrow$ minimum

(and these are the only two turning points).

[The shape of a quintic means that, if $f'(x) = 0$ and $f''(x) \neq 0$ for $x = -2$ and $x = 2$, then there would have to be a maximum at $x = -2$ and a minimum at $x = 2$.]

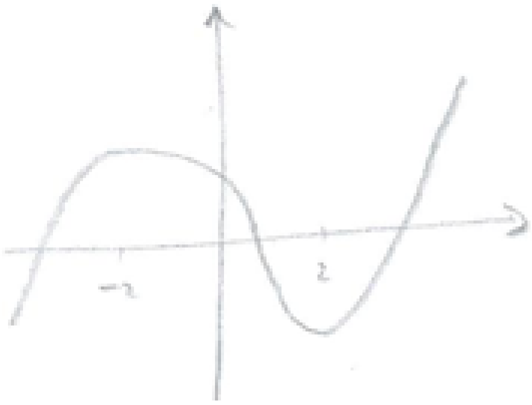
$f(-2) = 3(-32) - 10(-8) - 120(-2) + 30$

$= -96 + 80 + 240 + 30 > 0$

and $f(2) = 3(32) - 10(8) - 120(2) + 30$

$= 96 - 80 - 240 + 30 < 0$

so that the graph of $f(x)$ has the shape shown in the diagram below, and therefore there are 3 real roots of $f(x) = 0$



Answer : C

TMUA 2020, P1, Q10 [Transformations]

10 The following sequence of transformations is applied to the curve $y = 4x^2$

1. Translation by $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$

2. Reflection in the x-axis

3. Stretch parallel to the x-axis with scale factor 2

What is the equation of the resulting curve?

A $y = -x^2 + 12x - 31$

B $y = -x^2 + 12x - 41$

C $y = x^2 + 12x + 31$

D $y = x^2 + 12x + 41$

E $y = -16x^2 + 48x - 31$

F $y = -16x^2 + 48x - 41$

G $y = 16x^2 - 48x + 31$

H $y = 16x^2 - 48x + 41$

Solution

Translation by $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$: $y = 4x^2 \to y = 4(x-3)^2 - 5$

Reflection in the x-axis:

$y = 4(x-3)^2 - 5 \to y = -[4(x-3)^2 - 5]$

Stretch parallel to the x-axis with scale factor 2:

$y = -[4(x-3)^2 - 5] \to y = -[4\left(\frac{x}{2} - 3\right)^2 - 5]$

$= -x^2 + 12x - 31$

Answer: A

TMUA 2021, P2, Q6 [Polynomials]

6 Consider the following two statements about the polynomial $f(x)$:

P: $f(x) = 0$ for exactly three real values of x

Q: $f'(x) = 0$ for exactly two real values of x

Which one of the following is correct?

A P is **necessary** but **not sufficient** for Q.

B P is **sufficient** but **not necessary** for Q.

C P is **necessary and sufficient** for Q.

D P is **not necessary** and **not sufficient** for Q.

Solution

[It is assumed that "3 real roots" means "3 distinct real roots".]

Consider the following 4 examples:

(i) Cubic with 3 real roots, and therefore 2 stationary points (both P & Q true).

(ii) Polynomial with 3 real roots, but more than 2 stationary points (P true, but Q not true).

(iii) Cubic with 1 real root, and 2 stationary points (P not true, but Q true).

(iv) Cubic with 1 real root, and no stationary points (P not true, and Q not true).

So $P \nRightarrow Q$ & $Q \nRightarrow P$.

A can be written as: $Q \Rightarrow P$ & $P \nRightarrow Q$. So, as $Q \Rightarrow P$ is not true, A is not correct.

B can be written as: $P \Rightarrow Q$ & $Q \nRightarrow P$. So, as $P \Rightarrow Q$ is not true, B is not correct.

C can be written as: $Q \Rightarrow P$ & $P \Rightarrow Q$. So, as $Q \Rightarrow P$ is not true (for example), C is not correct.

D can be written as: $Q \nRightarrow P$ & $P \nRightarrow Q$. So, as both statements are true, D is correct.

Answer: D

TMUA 2021, P2, Q8 [Polynomials]

8 Consider the following statement about the polynomial $\mathrm{p}(x)$, where a and b are real numbers with $a < b$:

(∗) There exists a number c with $a < c < b$ such that $\mathrm{p}'(c) = 0$.

Which one of the following is true?

A The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **necessary and sufficient** for (∗)

B The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **necessary** but **not sufficient** for (∗)

C The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **sufficient** but **not necessary** for (∗)

D The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **not necessary** and **not sufficient** for (∗)

Solution

$p(a) = p(b)$ is a sufficient condition: the curve $y = p(x)$ is either a straight line, in which case $p'(x) = 0$ for all x between a & b; or $y = p(x)$ rises above $p(a)$ to achieve a maximum, before falling to $p(b) = p(a)$ (possibly after one or more minima or maxima); or $y = p(x)$ falls below $p(a)$ to achieve a minimum. In each of these cases (*) is satisfied.

$p(a) = p(b)$ isn't a necessary condition: a maximum or minimum could exist between a & b when $p(a) \neq p(b)$

Answer: C

MAT 2010, Q1/E [Logarithms]

E. Which is the largest of the following four numbers?

(a) $\log_2 3$, (b) $\log_4 8$, (c) $\log_3 2$, (d) $\log_5 10$.

Solution

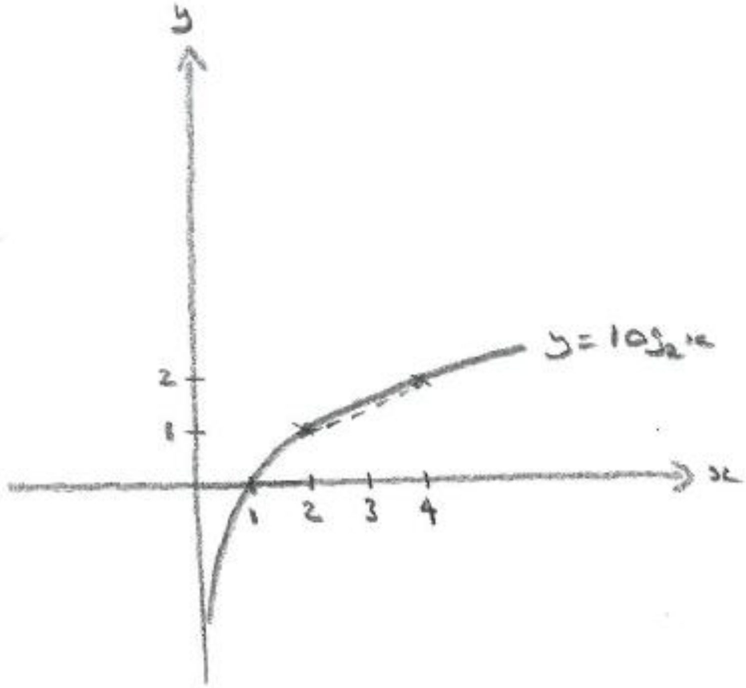
Let $log_2 3 = a$ etc

Note first of all that

$1 < a < 2, 1 < b < 2,\ 0 < c < 1$ & $1 < d < 2$, so that (c) can be eliminated.

Also $b = log_4(2^3) = 3log_4 2 = 3\left(\frac{1}{2}\right) = 1.5$

Now, from the diagram below we see that $a = log_2 3 > 1.5$



So that (b) can be eliminated.

Then $log_5 10 = log_5(5 \times 2) = log_5 5 + log_5 2 = 1 + log_5 2$

$< 1 + log_5\sqrt{5} = 1.5$, so that $d < 1.5 < a$

and **the answer is therefore (a).**

TMUA 2023, P2, Q6 [Algebra]

6 Consider the following equation where a is a real number and $a > 1$:

$$(*) \quad a^x = x$$

Which of the following equations must have the same number of real solutions as $(*)$?

I $\log_a x = x$

II $a^{2x} = x^2$

III $a^{2x} = 2x$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Solution

$a^x = x$ is equivalent to $log_a x = x$, so I has the same number of sol'ns as (*)

$a^x = x \Rightarrow a^{2x} = x^2$, but $a^{2x} = x^2 \Rightarrow a^x = x \ or \ a^x = -x$

[so potentially there may be extra sol'ns to II where $x < 0$]

Consider $a = 2$, and let $y = 2^x + x$. We want to see if there are any (negative) values of x for which $y = 0$:

When $x = -1, y = -\frac{1}{2} < 0$, and when $x = -\frac{1}{2}$, $y = \frac{1}{\sqrt{2}} - \frac{1}{2}$

$= \frac{\sqrt{2}-1}{2} > 0$

So, as $2^x + x$ is a continuous function, the change of sign means That there is a solution to $2^x + x = 0$, and thus to $a^x = -x$.

So II has more sol'ns than (*).

Finally, there is a 1-1 correspondence between sol'ns of $a^x = x$ and sol'ns of $a^{2x} = 2x$ (setting $y = 2x$), so that III has the same number of sol'ns as (*).

Answer : F

TMUA 2023, P2, Q7 [Proof]

7 The graph of the line $ax + by = c$ is drawn, where a, b and c are real non-zero constants.

Which one of the following is a **necessary** but **not sufficient** condition for the line to have a positive gradient **and** a positive y-intercept?

A $\frac{c}{b} > 0$ and $\frac{a}{b} < 0$

B $\frac{c}{b} < 0$ and $\frac{a}{b} > 0$

C $a > b > c$

D $a < b < c$

E a and c have opposite signs

F a and c have the same sign

Solution

For $ax + by = c$, a positive gradient means that $-\frac{a}{b} > 0$, or $\frac{a}{b} < 0$,

and a positive y-intercept means that $\frac{c}{b} > 0$

Let (*) be the situation where both the gradient and the y-intercept are positive.

Thus, A is equivalent to (*); ie A is a necessary and sufficient condition for (*).

B: $\frac{a}{b} > 0$ is not a necessary condition

C: Let $a = -1, b = 2$ & $c = 3$, so that

$\frac{a}{b} = -\frac{1}{2} < 0$ and $\frac{c}{b} = \frac{3}{2} > 0$, and hence (*) is satisfied

But it is not true that $a > b > c$, so that this is not a necessary condition for (*).

D: Now let $a = -1, b = 3$ & $c = 2$, so that

$\frac{a}{b} = -\frac{1}{3} < 0$ and $\frac{c}{b} = \frac{2}{3} > 0$, and hence (*) is satisfied

But it is not true that $a < b < c$, so that this is not a necessary condition for (*).

For E & F: Suppose that $b > 0$. Then (*) $\Rightarrow \frac{a}{b} < 0 \Rightarrow a < 0$

and $\frac{c}{b} > 0 \Rightarrow c > 0$

If instead $b < 0$. Then (*) $\Rightarrow \frac{a}{b} < 0 \Rightarrow a > 0$

and $\frac{c}{b} > 0 \Rightarrow c < 0$

Thus (as $b \neq 0$), a & c must be of opposite sign; ie this is a necessary condition.

So F can be ruled out.

By elimination, we can conclude that E is the correct answer.

[E is not sufficient: consider the case $a = 1, b = 1, c = -1$, where $\frac{a}{b} > 0$, so that (*) is not satisfied.]

Answer : E

TMUA 2023, P2, Q13 [Proof]

13 Let x be a real number.

Which **one** of the following statements is a **sufficient** condition for **exactly** three of the other four statements?

A $x \geq 0$

B $x = 1$

C $x = 0$ **or** $x = 1$

D $x \geq 0$ **or** $x \leq 1$

E $x \geq 0$ **and** $x \leq 1$

Solution

[It may be worth starting at C, as the examiners could well be expecting most candidates to start at A, or E!]

$C \Rightarrow A, C \nRightarrow B, C \Rightarrow D$ & $C \Rightarrow E$ so C is the correct answer

[$A \nRightarrow B$ & $A \nRightarrow C$, so A is ruled out]

[$B \Rightarrow A, B \Rightarrow C, B \Rightarrow D$ & $B \Rightarrow E$, so B is not the answer]

[$D \nRightarrow A, D \nRightarrow B$, so D is ruled out]

[$E \Rightarrow A, E \nRightarrow B, E \nRightarrow C$, so E is ruled out]

Answer : C

TMUA 2023, P2, Q16 [Proof]

16 A sequence is defined by:

$$u_1 = a$$
$$u_2 = b$$
$$u_{n+2} = u_n + u_{n+1} \quad \text{for } n \geq 1$$

where a and b are positive integers. The highest common factor of a and b is 7.

Which of the following statements **must** be true?

I u_{2023} is a multiple of 7

II If u_1 is not a factor of u_2, then u_1 is not a factor of u_n for any $n > 1$

III The highest common factor of u_1 and u_5 is 7

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Solution

As 7 is a divisor of $u_1 = a$ & $u_2 = b$, it is a divisor of $u_3 = u_1 + u_2$, and of $u_4 = u_2 + u_3$ etc; so that 7 is a divisor of u_{2023};

ie statement I is true

$A \Rightarrow B$ is equivalent to $B' \Rightarrow A'$ [consider Venn diagram, where $A \subset B$]

So Statement II is equivalent to:

"If u_1 is a factor of u_n for some $n > 1$,

then u_1 is a factor of u_2" (*)

Consider $u_1 = 2, u_2$=3; so that $u_3 = 5, u_4 = 8$

Then (*) doesn't hold (and we can also see directly that Statement II isn't true).

For Statement III:

Now the HCF of a & b is 7, so that $a = 7m$ & $b = 7n$, where the HCF of the positive integers m & n is 1. Then $u_3 = a + b$,

$u_4 = (a + b) + b$ and $u_5 = (a + 2b) + (a + b) = 2a + 3b$

[On account of the presence of $3b$ here:]

Consider the case of $a = 21$ & $b = 7$. Then $u_5 = 2(21) + 3(7)$

$= 21(2 + 1)$, and as u_1 & u_5 have a HCF of 21, this provides a counterexample to Statement III, which is therefore not true.

Thus only Statement I is true.

Answer : B

TMUA 2023, P2, Q18 [Polynomials]

18 The equation $x^4 + bx^2 + c = 0$ has four distinct real roots **if and only if** which of the following conditions is satisfied?

A $b^2 > 4c$

B $b^2 < 4c$

C $c > 0$ and $b > 2\sqrt{c}$

D $c > 0$ and $b < -2\sqrt{c}$

E $c < 0$ and $b < 0$

F $c < 0$ and $b > 0$

Solution

If $b^2 > 4c$, then the equation $y^2 + by + c = 0$ (*) has the distinct roots $y = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

Then, in order for the equation $x^4 + bx^2 + c = 0$ to have 4 distinct roots, both of the roots of (*) must be positive, so that

$-b - \sqrt{b^2 - 4c} > 0$; ie $\sqrt{b^2 - 4c} < -b$,

which requires $b < 0$ and $c > 0$

Thus, sufficient and necessary conditions for 4 distinct roots are:

$b^2 > 4c, b < 0$ and $c > 0$,

or equivalently $c > 0, b < -2\sqrt{c}$

Answer : D

TMUA 2023, P2, Q19 [Polynomials]

19 In this question, $f(x)$ is a non-constant polynomial, and $g(x) = xf'(x)$

$f(x) = 0$ for exactly M real values of x.

$g(x) = 0$ for exactly N real values of x.

Which of the following statements is/are true?

I It is possible that $M < N$

II It is possible that $M = N$

III It is possible that $M > N$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Solution

Consider $f(x) = x^2 + x - 1$, so that $f(x) = 0$ has 2 distinct roots.

Then $g(x) = x(2x + 1) = 2x^2 + x$, so that $g(x) = 0$ has 2 distinct roots; ie $M = N$

Thus Statement II is true.

Instead, let $f(x) = x^2 + x + 1$, so that $f(x) = 0$ has no roots.

Then $g(x) = x(2x + 1) = 2x^2 + x$ again, so that $g(x) = 0$ has 2 distinct roots; ie $M < N$

Thus Statement I is true.

[If $f(x)$ is a quadratic, then we can see from the above that $g(x) = 0$ will always have 2 distinct roots; thus $M > N$ isn't possible in this situation.]

Considering the graph of $y = f(x)$, we see that, for there to be M roots, there must be at least $M - 1$ turning points. [Consider a cubic, for example.] But one of these turning points could occur when $x = 0$. For example, if $f(x) = x^2(x - 1) + a$, where $a > 0$ is sufficiently small for the graph of $y = f(x)$ to cross the x-axis 3 times.

Then $g(x) = x[2x(x - 1) + x^2] = x^2(3x - 2)$. So $M = 3$ and $N = 2$

Thus Statement III is also true.

Answer : H

TMUA 2023, P2, Q20 [Proof]

20 Let f be a polynomial with real coefficients.

The integral $I_{p,q}$ where $p < q$ is defined by

$$I_{p,q} = \int_p^q (f(x))^2 - (f(|x|))^2 \, dx$$

Which of the following statements must be true?

1 $I_{p,q} = 0$ **only if** $0 < p$

2 $f'(x) < 0$ for all x **only if** $I_{p,q} < 0$ for all $p < q < 0$

3 $I_{p,q} > 0$ **only if** $p < 0$

A none of them

B 1 only

C 2 only

D 3 only

E 1 and 2 only

F 1 and 3 only

G 2 and 3 only

H 1, 2 and 3

Solution

"Only if" is equivalent to "implies".

Let $f(x) = x^2$. Then $f(|x|) = f(x)$ for all x, and so the integrand of $I_{p,q}$ is zero for all x (and hence $I_{p,q} = 0$), regardless of the value of p. Thus $I_{p,q} = 0 \nRightarrow 0 < p$, and so Statement I is not true.

[It looks as if it would be difficult to prove the truth of Statement II, so it may be best to look for a counterexample first.]

Consider $f(x) = -x$ (so that $f'(x) < 0$ for all x).

Then the integrand of $I_{p,q}$ is $(-x)^2 - (-|x|)^2 = 0$ for all x, and hence $I_{p,q} = 0$. Thus Statement II is not true.

[If we were short of time, this would be a good one to guess!]

Re. Statement III, if $p \geq 0$, then $f(|x|) = f(x)$, and so the integrand of $I_{p,q}$ is zero, and therefore $I_{p,q} = 0$. Thus if $I_{p,q} > 0$, it follows that $p < 0$.

Thus Statement III is true.

[Alternatively, we can say that Statement III ($I_{p,q} > 0 \Rightarrow p < 0$) is equivalent to "$not\ (p < 0) \Rightarrow not(I_{p,q} > 0)$", or

$p \geq 0 \Rightarrow I_{p,q} \leq 0$ (*)

We have shown that $p \geq 0 \Rightarrow I_{p,q} = 0$, which means that (*) is true.]

Answer : D

TMUA 2021, P1, Q7 [Integration]

The function f is such that $f(0) = 0$, and $xf(x) > 0$ for all non-zero values of x.

It is given that

$$\int_{-2}^{2} f(x)\, dx = 4$$

and

$$\int_{-2}^{2} |f(x)|\, dx = 8$$

Evaluate

$$\int_{-2}^{0} f(|x|)\, dx$$

A -8

B -6

C -4

D -2

E 2

F 4

G 6

H 8

Solution

Let $I = \int_{-2}^{0} f(|x|)dx = \int_{-2}^{0} f(-x)dx$

Write $u = -x$, so that $I = \int_{2}^{0} f(u)(-1)du = \int_{0}^{2} f(x)dx$ (*)

As $xf(x) > 0, f(x) > 0$ when $x > 0$, and $f(x) < 0$ when $x < 0$.

Then $\int_{-2}^{2} |f(x)|dx = 8 \Rightarrow \int_{-2}^{0} -f(x)dx + \int_{0}^{2} f(x)dx = 8$ (1)

And $\int_{-2}^{2} f(x)dx = 4$, so that $\int_{-2}^{0} f(x)dx + \int_{0}^{2} f(x)dx = 4$ (2)

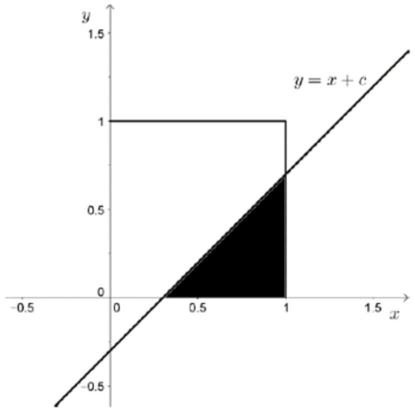
Adding (1) & (2): $2\int_{0}^{2} f(x)dx = 12$,

so that, from (*), $I = \int_{0}^{2} f(x)dx = 6$

Answer: G

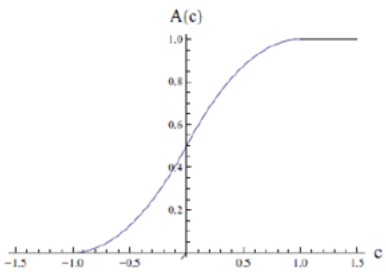
MAT 2012, Q1/D [Differentiation]

Shown below is a diagram of the square with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$ and the line $y = x + c$. The shaded region is the region of the square which lies below the line; this shaded region has area $A(c)$.

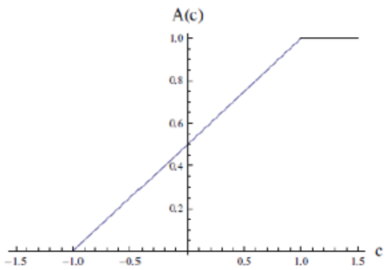


Which of the following graphs shows $A(c)$ as c varies?

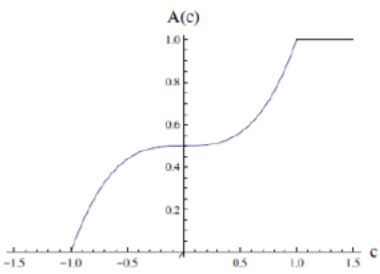
(a)



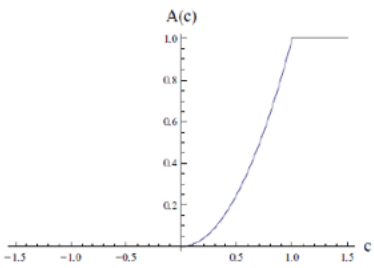
(b)



(c)



(d)



Solution

A(c) increases at its greatest rate when $c = 0$, and this agrees with (a) only.

So the answer is (a).

[Alternatively: $A(0) = 0.5$, so that (d) can be eliminated.

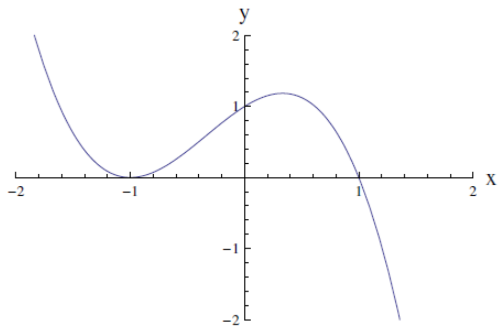
Then, for $c \leq 0$, $A(c) = \int_{-c}^{1} x + c\, dx = \left[\frac{1}{2}x^2 + cx\right]_{-c}^{1}$

$= \left(\frac{1}{2} + c\right) - \left(\frac{1}{2}c^2 - c^2\right) = \frac{1}{2}c^2 + c + \frac{1}{2}$

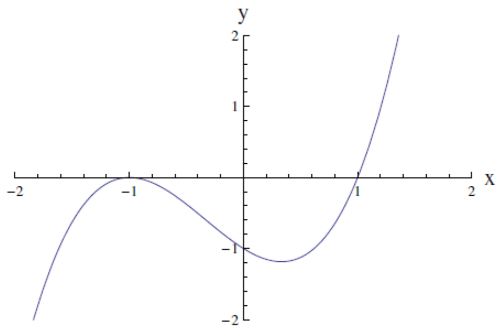
Option (b) is therefore eliminated, as it isn't a quadratic function for $c \leq 0$; whilst (c) is the wrong-shaped quadratic (being 'n-shaped', rather than 'u-shaped'). Also $A'(c) = c + 1$, so that $A'(0) = 1$, and this is inconsistent with (c), which shows a gradient of zero.]

MAT 2011, Q1/A [Cubics]

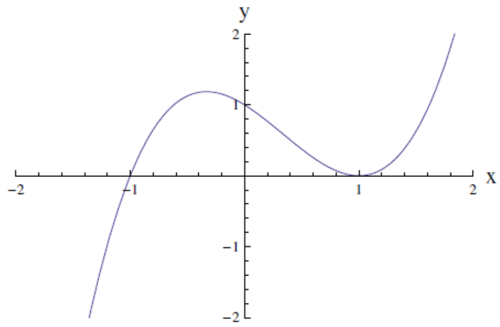
A. A sketch of the graph $y = x^3 - x^2 - x + 1$ appears on which of the following axes?



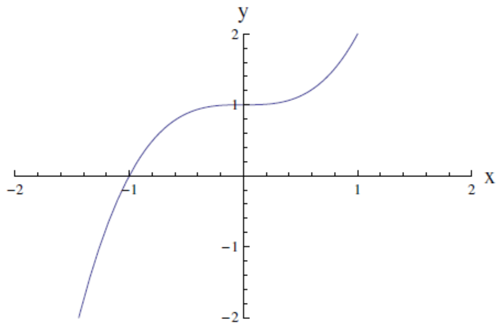
(a)



(b)



(c)



(d)

Solution

(a) starts in the wrong quadrant, and so can be eliminated.

If $f(x) = x^3 - x^2 - x + 1$,

$f'(x) = 3x^2 - 2x - 1 = (x - 1)(3x + 1)$

[Had $(x - 1)$ not been a factor, (c) could have been eliminated.]

Thus there is a stationary point at $x = 1$,

and so **the answer must be (c)**, by elimination.

[Alternative approaches:

(i) Consider x-coordinate of point of inflexion $(-\frac{b}{3a} = \frac{1}{3})$

(ii) Sum of roots is expected to be $-\frac{b}{a} = 1$, which rules out (a) & (b), and is consistent with (c). However, there are two (as yet unknown) complex roots for (d).]

TMUA 2021, P2, Q20 [Integration]

20 A sequence of functions f_1, f_2, f_3, ... is defined by

$$f_1(x) = |x|$$

$$f_{n+1}(x) = |f_n(x) + x| \quad \text{for } n \geq 1$$

Find the value of

$$\int_{-1}^{1} f_{99}(x)\,dx$$

A 0

B 0.5

C 1

D 49.5

E 50

F 99

G 99.5

H 100

Solution

If $x \geq 0$, $|x| = x$, and so $f_n(x) = nx$

If $x < 0$, $f_1(x) = -x$, $f_2(x) = |-x + x| = 0$,

$f_3(x) = |0 + x| = -x$, $f_4(x) = |-x + x| = 0$, and so on

So $f_{99}(x) = -x$ (when $x < 0$).

Hence $\int_{-1}^{1} f_{99}(x)\, dx = \int_{-1}^{0} -x\, dx + \int_{0}^{1} 99x\, dx$

$= \left[-\frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{99}{2}x^2\right]_{0}^{1}$

TMUA 2016, P2, Q4 [Logic]

5. TMUA 2016 Paper 2 Q4

Five sealed urns, labelled P, Q, R, S, and T, each contain the **same** (non-zero) number of balls. The following statements are attached to the urns.

Urn P	This urn contains one or four balls.
Urn Q	This urn contains two or four balls.
Urn R	This urn contains more than two balls and fewer than five balls.
Urn S	This urn contains one or two balls.
Urn T	This urn contains fewer than three balls.

Exactly one of the urns has a true statement attached to it.

Which urn is it?

A Urn P
B Urn Q
C Urn R
D Urn S
E Urn T

Solution

If n is the number of balls in all the urns:

Method 1

Considering $n = 1, 2, 3, \ldots$ in turn, we can see which of the statements P,Q, R, … are consistent with the value for n:

n	1	2	3
P	Y	X	X
Q	X	Y	X
R	X	X	Y
S	Y	Y	X
T	n/a	n/a	X

So the answer must be urn R.

Method 2

Suppose that P is telling the truth ("$P = 1$").

Then if n is the number of balls in all the urns, $n = 1 \; or \; 4$.

Then $Q = 0$, so that $n \neq 4$, and therefore $n = 1$.

Now, $R = 0$, so that n is neither 3 nor 4; which is consistent with $n = 1$.

And $S = 0$, so that n is neither 1 nor 2, which contradicts $n = 1$.

Therefore $\boldsymbol{P = 0}$, and n is neither 1 nor 4.

Suppose now that $Q = 1$, so that $n = 2$ or 4. Then, as n is neither 1 nor 4, it follows that $n = 2$.

Then $R = 0$, so that n is neither 3 nor 4; which is consistent with $n = 2$.

And $S = 0$, so that n is neither 1 nor 2, which contradicts $n = 1$.

Therefore $\boldsymbol{Q = 0}$, and n is neither 2 nor 4. And (as $P = 0$), n is neither 1 nor 4. Thus, n is neither 1 nor 2 nor 4.

Suppose now that $R = 1$, so that $n = 3$ or 4. Then (from the previous statement), it follows that $n = 3$.

Now, $S = 0$, so that n is neither 1 nor 2, which is consistent with $n = 3$.

And $T = 0$, so that $n \geq 3$.

Thus $n = 3$ is a possible solution, with $R = 1$, and we infer from the question that it is possible to deduce which urn has the true statement, so that the answer must be urn R.

Answer: C

TMUA 2016, P2, Q10 [Proof]

7. TMUA 2016 Paper 2 Q10

In this question x and y are non-zero real numbers.

Which one of the following is **sufficient** to conclude that $x < y$?

A $x^4 < y^4$

B $y^4 < x^4$

C $x^{-1} < y^{-1}$

D $y^{-1} < x^{-1}$

E $x^{\frac{3}{5}} < y^{\frac{3}{5}}$

F $y^{\frac{3}{5}} < x^{\frac{3}{5}}$

Solution

To prove that $P \Rightarrow Q$ (ie that the truth of P is sufficient to conclude that Q is true), we could instead prove that $Q' \Rightarrow P'$.

(consider P being "lives in London" and Q being "lives in England")

Considering A:

Result to prove or disprove: $x \geq y \Rightarrow x^4 \geq y^4$

Case 1: $y \geq 0$ Then $x \geq y \Rightarrow x^4 \geq y^4$

Case 2: $y < 0, x < 0$ Then $x \geq y \Rightarrow x^4 < y^4$

Thus, if $y < 0, x < 0$, we can't say that $x \geq y \Rightarrow x^4 \geq y^4$,

and so A isn't sufficient to conclude that $x < y$.

[Alternatively, counter-example is $x = 1, y = -2$]

Considering B:

Result to prove or disprove: $x \geq y \Rightarrow y^4 \geq x^4$

But if $x > y > 0$, then it is not true that $y^4 \geq x^4$.

Considering C:

Result to prove or disprove: $x \geq y \Rightarrow \frac{1}{x} \geq \frac{1}{y}$

Counter-example: $x = 2, y = 1$

Considering D:

Result to prove or disprove: $x \geq y \Rightarrow \frac{1}{y} \geq \frac{1}{x}$

Counter-example: $x = 1, y = -1$

Considering E:

Result to prove or disprove: $x \geq y \Rightarrow x^{\frac{3}{5}} \geq y^{\frac{3}{5}}$

First of all note that $x^{\frac{1}{5}}$ is defined for negative x (eg for $x = -32$,

$(-32)^{\frac{1}{5}} = -2$, as $(-2)^5 = -32$).

Now, $y = x^3$ is an increasing function, so that $x \geq y \Rightarrow x^3 \geq y^3$.

And $y = x^5$ is also an increasing function, so that

$x^3 \geq y^3 \Rightarrow x^{\frac{3}{5}} \geq y^{\frac{3}{5}}$

So E is the correct answer.

[For F, the result to prove or disprove is: $x \geq y \Rightarrow y^{\frac{3}{5}} \geq x^{\frac{3}{5}}$

Counter-example is $x = 1, y = -32$]

Answer: E

TMUA Specimen P2, Q18 [Statistics]

18. A group of five numbers are such that:

- their mean is 0
- their range is 20

What is the largest possible median of the five numbers?

A 0

B 4

C $4\frac{1}{2}$

D $6\frac{1}{2}$

E 8

F 20

Solution

Let the 5 numbers be $m-a-b, m-a, m, m+c, m+c+d$

where $a, b, c \; \& \; d$ $are\ all\ \geq 0$

Then $(m-a-b)+(m-a)+m+(m+c)+(m+c+d)=0$ (1)

and $(m+c+d)-(m-a-b)=20$ (2)

From (2), $c+d+a+b=20$,

and then from (1):

$(m-a-b)+(m-a)+m+(m+c)+(m+20-a-b)=0,$

So that $5m=3a+2b-c-20$

$=2(c+d+a+b)+a-3c-2d-20$

$=20+a-3c-2d$

$=20+(20-c-d-b)-3c-2d$

[aiming for a form where the letters all have negative signs]

$=40-4c-3d-b$

and this is maximised when $b=c=d=0$, so that $a=20$

(as $c+d+a+b=20$) and $m=8$

[Then the 5 numbers are $-12,-12,8,8,8$]

Answer: E

Approach 2 (Trial & Improvement)

One sol'n (satisfying the 2 conditions) is -10, 0, 0, 0, 10,

So that M (the largest possible value for the median) ≥ 0 (as indicated by the multiple choice options).

The median can then be increased to 5 without changing the 1st and last values, giving -10, -10, 5, 5, 10 (the 4th value has to be at least 5, and if equals 5, then the 2nd value has to be -10, to maintain the mean of 0).

The median can then be increased to 8 (as suggested by the MC options) by increasing the 4th & 5th values to 8, which allows the 1st value to be lowered to -12, which accommodates a 2nd value of -12 (needed to maintain the mean of 0).

Thus, $M \geq 8$

But $M = 20$ isn't possible, as the 4th & 5th values would then have to be at least 20, forcing the 1st value to be at least 0 (for the range to be 20). But this would give a mean > 0.

So $8 \leq M < 20$, and from the MC options available, M must be 8.