TMUA: Algebra – Ideas & Exercises (12 pages; 20/12/24)

Standard problem solving device

- Express the given information in the form of one or more equations (or sometimes inequalities)

- Possibly using a standard result, such as Pythagoras' theorem or Hooke’s law

- Then solve the equations

Rearranging an equation to get zero on one side

Example: Show that $\frac{1-tan^2\theta}{1+tan^2\theta} \equiv cos2\theta$

Solution

Result to prove: $\frac{1-tan^2\theta}{1+tan^2\theta} - cos2\theta \equiv 0$

LHS $= \frac{1-tan^2\theta}{sec^2\theta} - (cos^2\theta - sin^2\theta)$

$= cos^2\theta(1 - tan^2\theta) - cos^2\theta + sin^2\theta$

$= -cos^2\theta.\frac{sin^2\theta}{cos^2\theta} + sin^2\theta$

$= -sin^2\theta + sin^2\theta = 0$

Solve the equation $x - \sqrt{x} = 6$

Solve the equation $x - \sqrt{x} = 6$

Solution

Let $f(x) = x - \sqrt{x} - 6$

$f(x) = 0 \Rightarrow x - 6 = \sqrt{x}$

$\Rightarrow (x-6)^2 = x$, but this may include spurious solutions

[of $x - 6 = -\sqrt{x}$]

$\Rightarrow x^2 - 13x + 36 = 0$

$\Rightarrow (x-9)(x-4) = 0$

$\Rightarrow x = 9 \; or \; x = 4$

$f(9) = 0$ & $f(4) = -4$

Thus the only solution is $x = 9$

[Let $g(x) = x + \sqrt{x} - 6 = 0$

Then $g(x) = 0 \Rightarrow (x-6)^2 = x$ as well

$g(9) \neq 0, and \; g(4) = 0$]

Alternatively: Let $y = \sqrt{x}$, so that

$x - \sqrt{x} - 6 = 0 \Rightarrow y^2 - y - 6 = 0$

$\Rightarrow (y+2)(y-3) = 0$

$\Rightarrow y = -2 \left(\text{reject as } \sqrt{\text{x}} \text{ must be} \geq 0\right) or \; y = 3$

Solve the equation $\sqrt{2x+3}+\sqrt{x+1}=\sqrt{7x+4}$

Solve the equation $\sqrt{2x+3} + \sqrt{x+1} = \sqrt{7x+4}$

Solution

$\sqrt{2x+3} + \sqrt{x+1} = \sqrt{7x+4}$ (*)

$\Rightarrow (2x+3) + 2\sqrt{(2x+3)(x+1)} + (x+1) = 7x+4$

(incl. possible spurious sol'ns)

$\Rightarrow 2\sqrt{(2x+3)(x+1)} = 4x$

$\Rightarrow (2x+3)(x+1) = 4x^2$

$\Rightarrow 2x^2 - 5x - 3 = 0$

$\Rightarrow (2x+1)(x-3) = 0$

$\Rightarrow x = -\frac{1}{2}\ or\ 3$

But only $x = 3$ satisfies (*)

[$x = -\frac{1}{2}$ is a sol'n of $2\sqrt{(2x+3)(x+1)} = -4x$]

Factorise $x^3 - y^3$ and $x^3 + y^3$ in the form

$(x - y)(\ldots)$ or $(x + y)(\ldots)$

Factorise $x^3 - y^3$ and $x^3 + y^3$ in the form

$(x - y)(\dots)$ or $(x + y)(\dots)$

Solution

Write $f(x) = x^3 - y^3$

As $f(y) = 0$ for all y, $(x - y)$ is a factor,

and $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

But $f(-y) \neq 0$ (unless $y = 0$), so that $(x + y)$ isn't a factor.

Similarly, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$,

But $(x - y)$ isn't a factor.

Repeat for a general even number n.

Solution

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

Or $(x + y)(x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1})$

[Consider $x^2 - y^2 = (x - y)(x + y)$]

And $x^n + y^n > 0$ (unless $x = y = 0$), and so neither $(x + y)$ nor $(x - y)$ are factors.

Expand the following:

(i) $(a+b+c)^2$

(ii) $(a+b+c)^3$

Solution

(i) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$

(ii) $(a+b+c)^3 = (a^3 + b^3 + c^3) + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 6abc$

[Consider symmetry and the number of combinations of each type of term. For example, there are 3 ways of creating an a^2b term: 3[number of ways of choosing the b]$\times$ 1[number of ways of choosing two as from the remaining 2 brackets].]

Avoiding division by zero

[division by zero is 'undefined']

Example: Solve $tan\theta = sin\theta$, for $0 \leq \theta < 360$

Solution

$tan\theta = sin\theta \Rightarrow \frac{sin\theta}{cos\theta} = sin\theta$

Approach 1

Case 1: $sin\theta \neq 0$

$\Rightarrow \frac{1}{cos\theta} = 1$ [both sides can only be divided by $sin\theta$ if $sin\theta \neq 0$]

$\Rightarrow 1 = cos\theta$

$\Rightarrow \theta = 0$

Case 2: $sin\theta = 0$ (as this satisfies $\frac{sin\theta}{cos\theta} = sin\theta$)

$\Rightarrow \theta = 0 \ or \ 180$

Conclusion

$\theta = 0 \ or \ 180$

Approach 2

$tan\theta = sin\theta \Rightarrow \frac{sin\theta}{cos\theta} - sin\theta = 0$

$\Rightarrow sin\theta \left(\frac{1}{cos\theta} - 1\right) = 0$

$\Rightarrow sin\theta = 0 \ (A) \ or \ \frac{1}{cos\theta} - 1 = 0 \ (B)$

Then as for Approach 1.