TMUA – Basic Ideas & Exercises (20 pages; 18/11/24)

(i) Does $\sqrt{4}$ equal 2 or ± 2? (ii) Simplify $\sqrt{x^2}$

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Solution

(i) 2 (ii) $|x|$

Simplify the following:

(i) $27^{-\frac{2}{3}}$ (ii) $\cos(-210°)$ (iii) $log_4\left(\frac{1}{64}\right)$

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Solution

(i) $\frac{1}{27^{\frac{2}{3}}} = \frac{1}{\left(27^{\frac{1}{3}}\right)^2} = \frac{1}{3^2} = \frac{1}{9}$

(ii) $\cos(-210°) = \cos(210°) = \cos(360° - 210°) = \cos(150°) = -\frac{\sqrt{3}}{2}$

(iii) $log_4\left(\frac{1}{64}\right) = log_4(4^{-3}) = -3$

Prove that $sin^2\theta + cos^2\theta = 1$

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Solution

Apply Pythagoras to a right-angled triangle with sides cosθ, sinθ & 1.

Find the turning point of the graph of $y = (x - 1)(x + 2)$

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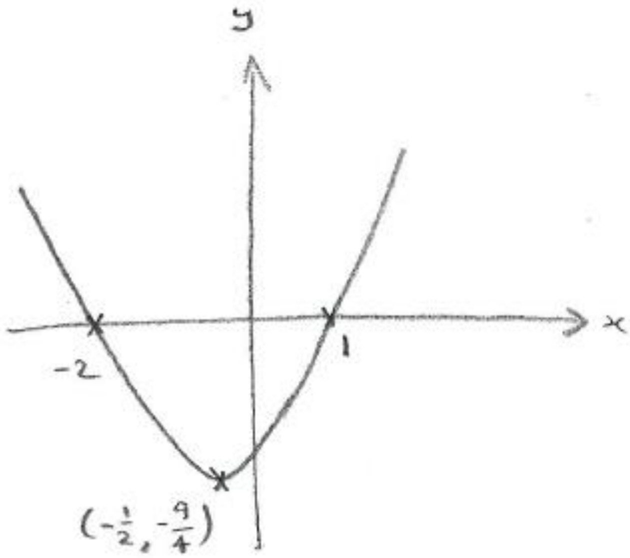
Solution



Due to the symmetry of the curve about the vertical line through the turning point, the x-coordinate of the turning point will be $\frac{1}{2}(-2+1) = -\frac{1}{2}$

Then the y-coordinate is $= \left(-\frac{1}{2} - 1\right)\left(-\frac{1}{2} + 2\right) = \left(\frac{-3}{2}\right)\left(\frac{3}{2}\right) = -\frac{9}{4}$

Alternatively, we can complete the square:

$$(x-1)(x+2) = x^2 + x - 2 = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - 2 = \left(x + \frac{1}{2}\right)^2 - \frac{9}{4}$$

giving the turning point of $\left(-\frac{1}{2}, -\frac{9}{4}\right)$

Composite transformations required to obtain

$y = \sin(2x + 60)$ from $y = sinx$?

Either (a) stretch by scale factor $\frac{1}{2}$ in the x direction, to give

$y = \sin(2x)$, and then translate by $\begin{pmatrix} -30 \\ 0 \end{pmatrix}$, to give

$y = \sin(2[x + 30]) = \sin\,(2x + 60)$

or (b) translate by $\begin{pmatrix} -60 \\ 0 \end{pmatrix}$, to give $y = \sin(x + 60)$, and then

stretch by scale factor $\frac{1}{2}$ in the x direction, to give

$y = \sin(2x + 60)$

(It is perhaps more awkward to produce a sketch by method (b).)

Express $-cos\theta$ in the form $cos\alpha$ (where α is to be found in terms of θ), using an algebraic method.

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Solution

$-cos\theta = -\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\theta - \frac{\pi}{2}\right)$

$= \cos\left(\frac{\pi}{2} - \left[\theta - \frac{\pi}{2}\right]\right) = \cos(\pi - \theta)$ (or $\cos(3\pi - \theta)$ etc)

Alternatively, $-cos\theta = -\cos(-\theta) = -\sin\left(\frac{\pi}{2} - [-\theta]\right)$

$= \sin\left(-\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - \left[-\frac{\pi}{2} - \theta\right]\right) = \cos(\pi + \theta)$

(or $\cos(3\pi + \theta)$ etc)

Formula for (i) $\sum_{r=1}^{n} r$ (ii) $\sum_{r=1}^{n} r^2$ (iii) $\sum_{r=1}^{n} r^3$

Solution

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

(i) Expand $(a+b+c)^2$ (ii) Expand $(a+b+c)^3$

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Solution

(i) $(a+b+c)^2 = (a^2+b^2+c^2) + 2(ab+ac+bc)$

(ii) $(a+b+c)^3 = (a^3+b^3+c^3)$

$+3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)$

$+6abc$

Factorise $15x^2 + 34x + 16$

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Solution

We want A and B such that $A + B = 34$ and $AB = (15)(16) = 240$

Again, the factorisation of 240 is $2^4 \times 3 \times 5$

Starting with |A| and |B| close to each other:

eg $A = 15,\ B = 16 \Rightarrow A + B = 31$

$A = 16,\ B = 15 \Rightarrow A + B = 31$ (ie no change)

$A = 20,\ B = 12 \Rightarrow A + B = 32$ (ie moving in the right direction)

$A = 24,\ B = 10 \Rightarrow A + B = 34$

Note: $A = 15, 12, 10$ also leads to a solution.

Then we have $(15x^2 + 24x) + (10x + 16\ )$

and $3x(5x + 8) + 2(5x + 8) = (3x + 2)(5x + 8)$

How many solutions are there to $x^3 - 6x^2 + 9x + 2 = 0$?

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Solution

$x^3 - 6x^2 + 9x + 2 = 0 \Leftrightarrow x(x^2 - 6x + 9) = -2$

$\Leftrightarrow x(x-3)^2 = -2$

So one solution, from graph of $y = x(x-3)^2$