Systems of Forces (5 pages; 30/1/17)

(1) Equivalent systems of forces

Two systems of forces are said to be equivalent if:

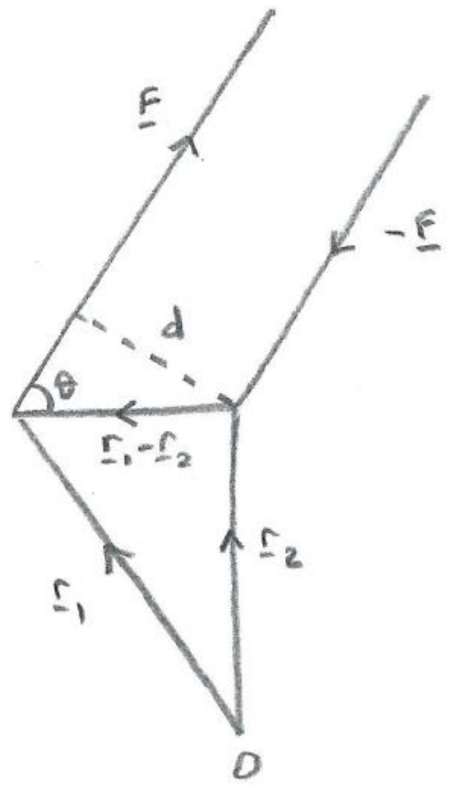
(i) their vector sums are equal, and

(ii) the vector sums of their moments about all points are equal

In fact, if the vector sums of their moments about one point are equal, then (ii) will follow.

Note: This applies to forces in 3D, as well as 2D.

(2) Couples



Consider a pair of forces $\boldsymbol{F}$ and $-\boldsymbol{F}$, passing through the points with position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, such that the lines of action of $\boldsymbol{F}$ and $-\boldsymbol{F}$

are a distance d apart (see diagram above). The vector sum of the moments of $\boldsymbol{F}$ and $-\boldsymbol{F}$ about O is

$$\boldsymbol{G} = (\boldsymbol{r_1} \times \boldsymbol{F}) + (\boldsymbol{r_2} \times -\boldsymbol{F}) = (\boldsymbol{r_1} - \boldsymbol{r_2}) \times \boldsymbol{F}$$

which has magnitude $|\boldsymbol{r_1} - \boldsymbol{r_2}|Fsin\theta = Fd$

and direction perpendicular to both **F** and $\boldsymbol{r_1} - \boldsymbol{r_2}$.

The system consisting of $\boldsymbol{F}$ and $-\boldsymbol{F}$ is termed a couple. The moment of the couple (ie $(\boldsymbol{r_1} - \boldsymbol{r_2}) \times \boldsymbol{F}$) has a magnitude and direction independent of the Origin.

Unlike the moment of a force, there is no particular axis about which a couple can be said to turn (ie a couple is not a **localised** vector). [A force is a localised vector, as it has a line of action associated with it.]

(3) Reduction of a system of forces to a single force (acting through an arbitrary point), together with a couple

Let $\boldsymbol{F}_1, \boldsymbol{F}_2, \dots, \boldsymbol{F}_n$ be a system of forces, passing through the points with position vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n$. Let $\boldsymbol{R} = \sum \boldsymbol{F}_i$, and suppose that **R** acts through an arbitrary point Q with position vector $\boldsymbol{q}$.

Then let $\boldsymbol{G} = (\sum \boldsymbol{r}_i \times \boldsymbol{F}_i) + (\boldsymbol{q} \times -\boldsymbol{R})$.

G can be shown to be independent of the Origin, as follows:

Let the new Origin, O' have position $\boldsymbol{a}$ relative to O, so that

$\boldsymbol{r'}_i = \boldsymbol{r}_i + \boldsymbol{a}$ and $\boldsymbol{q}' = \boldsymbol{q} + \boldsymbol{a}$

Then $\boldsymbol{G}' = (\sum \boldsymbol{r'}_i \times \boldsymbol{F}_i) + (\boldsymbol{q}' \times -\boldsymbol{R})$

$= (\sum(\boldsymbol{r}_i + \boldsymbol{a}) \times \boldsymbol{F}_i) + ((\boldsymbol{q} + \boldsymbol{a}) \times -\boldsymbol{R})$

$= (\sum \boldsymbol{r}_i \times \boldsymbol{F}_i) + (\sum \boldsymbol{a} \times \boldsymbol{F}_i) + (\boldsymbol{q} \times -\boldsymbol{R}) + (\boldsymbol{a} \times -\boldsymbol{R})$

$= (\sum \boldsymbol{r}_i \times \boldsymbol{F}_i) + (\boldsymbol{q} \times -\boldsymbol{R}) + \boldsymbol{a} \sum \boldsymbol{F}_i + (\boldsymbol{a} \times -\boldsymbol{R})$

$= (\sum \boldsymbol{r}_i \times \boldsymbol{F}_i) + (\boldsymbol{q} \times -\boldsymbol{R}) + \boldsymbol{aR} - \boldsymbol{aR} \;\; = \boldsymbol{G}$

Because of this independence from the Origin, G can be considered as the moment of a couple (ie it is equivalent to a pair of equal and opposite forces with the same total moment) .

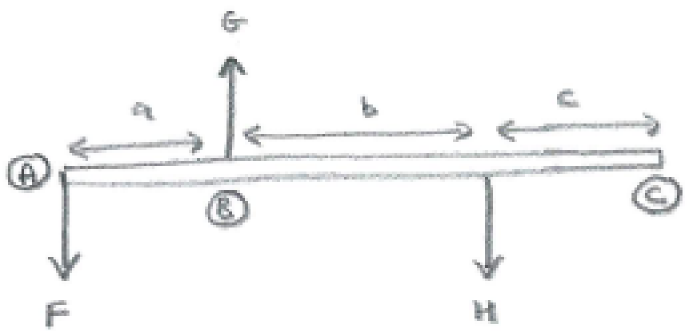
Then the system $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ is equivalent to the system consisting of **R**, together with the couple **G**, since the vector sum of the moments of the new system is $(\boldsymbol{q} \times \boldsymbol{R}) + \boldsymbol{G} = (\sum \boldsymbol{r}_i \times \boldsymbol{F}_i)$; ie the vector sum of the moments of the original system.

If Q is taken as the Origin, O (so that $\boldsymbol{q} = 0$), then the system is equivalent to a force $\boldsymbol{R}$ acting through O, together with a couple of moment $\sum \boldsymbol{r}_i \times \boldsymbol{F}_i$.

(4) Zero resultant

If the resultant force, $\boldsymbol{R} = 0$, then the system reduces to a couple of moment $\sum \boldsymbol{r}_i \times \boldsymbol{F}_i$, from (3). This has already been demonstrated to have the same value for any choice of Origin.

This last result can also be seen in the following simple 2D case, where $F + H - G = 0$



Net moment of forces

about A: $aG - (a + b)H = a(F + H) - (a + b)H = aF - bH$

about B: $aF - bH$

about C: $(a + b + c)F - (b + c)G + cH$

$= (a + b + c)F - (b + c)(F + H) + cH = aF - bH$

(5) Condition for a system of forces (in 3D) to reduce to a single force (with no couple)

From (3), the system $\boldsymbol{F}_1, \boldsymbol{F}_2, \dots, \boldsymbol{F}_n$ is equivalent to the system consisting of **R**, together with the couple

$\boldsymbol{G} = (\sum \boldsymbol{r}_i \times \boldsymbol{F}_i) + (\boldsymbol{q} \times -\boldsymbol{R})$

In order for **G** to be zero,

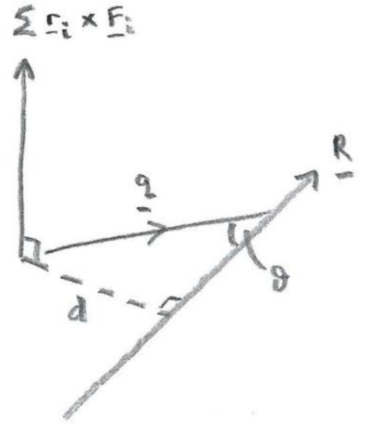
either (i) $\boldsymbol{q} = 0$ and $(\sum \boldsymbol{r}_i \times \boldsymbol{F}_i) = 0$

or (ii) $\sum \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{q} \times \boldsymbol{R}$ (A)

The position vector **q** can be chosen as appropriate. The only constraint is that $\sum \boldsymbol{r}_i \times \boldsymbol{F}_i$ is perpendicular to **R** (as required by (5)(A), since $\boldsymbol{q} \times \boldsymbol{R}$ will be perpendicular to **R**).

(5)(A) also implies that the moment (about the Origin) of $\boldsymbol{R}$ (whose line of action includes the point with position vector **q**) equals the sum of the moments of the $\boldsymbol{F}_i$.

In the diagram below, $\boldsymbol{R}$ is in the plane perpendicular to $\sum \boldsymbol{r}_i \times \boldsymbol{F}_i$, and the line of action of $\boldsymbol{R}$ is determined by the magnitude of $\sum \boldsymbol{r}_i \times \boldsymbol{F}_i$, which equals $|\boldsymbol{q} \times \boldsymbol{R}| = |q|R\sin\theta = dR$; thereby determining the distance d.



(6) Coplanar forces

If the $\boldsymbol{F}_i$ act in the same plane, then there are 3 possible situations:

(i) $\boldsymbol{R} = 0$ and $\sum \boldsymbol{r}_i \times \boldsymbol{F}_i = 0$; ie the system is in equilibrium

(ii) $\boldsymbol{R} = 0$ and $\sum \boldsymbol{r}_i \times \boldsymbol{F}_i \neq 0$; ie the system reduces to the couple $\boldsymbol{G} = (\sum \boldsymbol{r}_i \times \boldsymbol{F}_i)$, as in (4)

(iii) $\boldsymbol{R} \neq 0$ and, as in (5), the system reduces to a single force with no couple [$\sum \boldsymbol{r}_i \times \boldsymbol{F}_i$ will be perpendicular to **R**, since the $\boldsymbol{F}_i$ act in the same plane (and taking the Origin to be in that plane)]

References

"An introduction to Advanced Mechanics" - Quadling & Ramsay (1964)

"Mathematical Topics in Mechanics" - Dickie, Ovenstone, Smith & Waterston (1972)