Simplex Method - Exercises (Sol'ns) (7 pages; 14/8/19)

(1) Minimise $-3x + 2y + z$, subject to the following constraints:

$x + y - 4z \leq 4$

$-x + 3y + 2z \geq -2$

$x \geq 0, y \geq 0, z \geq 0$

Use the ordinary Simplex method to solve this problem.

Solution

Step 1: Rewrite the problem as

Maximise $P = 3x - 2y - z$,

subject to $x + y - 4z \leq 4$ and $x - 3y - 2z \leq 2$

Step 2: Create equations with slack variables [it is possible to skip this step, and go straight to the Simplex tableau]:

$P - 3x + 2y + z = 0$ (1)

$x + y - 4z + s_1 = 4$ (2)

$x - 3y - 2z + s_2 = 2$ (3)

Step 3: Represent the equations in a Simplex tableau:

P	x	y	z	s_1	s_2	value	row
1	-3	2	1	0	0	0	(1)
0	1	1	-4	1	0	4	(2)
0	(1)	-3	-2	0	1	2	(3)

Step 4: Choose x as the pivot column (as it has the largest negative coefficient in the objective row), and perform the ratio test to establish the pivot row.

As $\frac{2}{1} < \frac{4}{1}$, row 3 is the pivot row (indicated in the table above by the brackets - or circling if handwritten).

Step 5: Eliminate x from rows 1 and 2

As the coefficient of x for row 3 is already 1, no adjustment is needed for that row.

P	x	y	z	s_1	s_2	value	row
1	0	-7	-5	0	3	6	(4)=(1)+3(6)
0	0	(4)	-2	1	-1	2	(5)=(2)-(6)
0	1	-3	-2	0	1	2	(6)=(3)

Step 6: y now has the largest negative coefficient in the objective row, and as the coefficient of y in row 6 is negative, we can take row 5 as the pivot row.

Step 7: Eliminate y from rows 4 and 6

As the coefficient of y for row 5 is 4, we need to divide that row by 4 first.

P	x	y	z	s_1	s_2	value	row
1	0	0	$-8\frac{1}{2}$	$1\frac{3}{4}$	$1\frac{1}{4}$	$9\frac{1}{2}$	(7)=(4)+7(8)
0	0	1	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	(8)=(5)÷ 4

0	1	0	$-3\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$3\frac{1}{2}$	(9)=(6)+3(8)

Step 8: Although z has a negative coefficient in the objective row, the other coefficients of z are negative, and so no further progress can be made.

Hence the solution is: $x = 3\frac{1}{2}$, $y = \frac{1}{2}$, $z = 0, s_1 = 0, s_2 = 0, P = 9\frac{1}{2}$,

and hence the minimised value of $-3x + 2y + z$ is $-9\frac{1}{2}$

[Check: $x + y - 4z = 4 \leq 4$ and $-x + 3y + 2z = -2 \geq -2$]

(2) Maximise $5x - 2y + 4z$, subject to the following constraints:

$2x + y - z \leq 6$

$x - y + 2z \geq 5$

$3x + y - 7z \geq 4$

$x \geq 0, y \geq 0, z \geq 0$

Apply the 1st stage of the 2 Stage Simplex method, as far as establishing the pivot row for the 2nd time.

Solution

Step 1: Create equations with either slack variables, or surplus and artifical variables, as appropriate.

$P - 5x + 2y - 4z = 0$ (1)

$2x + y - z + s_1 = 6$ (2)

$x - y + 2z - s_2 + a_1 = 5$ (3)

$3x + y - 7z - s_3 + a_2 = 4$ (4)

Step 2: Let $A = a_1 + a_2 = (5 - x + y - 2z + s_2) + (4 - 3x - y + 7z + s_3)$

$= 9 - 4x + 5z + s_2 + s_3$

The 1st stage of the method is to minimise A.

Step 3: Represent the equations in a Simplex tableau:

A	P	x	y	z	s_1	s_2	s_3	a_1	a_2	value	row
1	0	4	0	-5	0	-1	-1	0	0	9	(1)
0	1	-5	2	-4	0	0	0	0	0	0	(2)
0	0	2	1	-1	1	0	0	0	0	6	(3)
0	0	1	-1	2	0	-1	0	1	0	5	(4)
0	0	(3)	1	-7	0	0	-1	0	1	4	(5)

Step 4: As we are minimising A, we look for large positive coefficients of variables in the 1st row (so that when the variable is maximised, it will reduce A as much as possible). Here there is no choice, and we take x as the pivot column, and perform the ratio test to establish the pivot row (ignoring any rows with negative coefficients of x).

row 3: $\frac{6}{2} = 3$; row 4: $\frac{5}{1} = 5$; row 5: $\frac{4}{3}$; so the pivot row is row 5 (indicated in the table above by the brackets - or circling if handwritten)

[Note: It is possible (though less usual) to maximise $-A$ instead.]

Step 5: Eliminate x from rows 1-4

A	P	x	y	z	s_1	s_2	s_3	a_1	a_2	value	row
1	0	0	$-\frac{4}{3}$	$\frac{13}{3}$	0	-1	$\frac{1}{3}$	0	$-\frac{4}{3}$	$\frac{11}{3}$	(6)=(1)−4(10)
0	1	0	$\frac{11}{3}$	$-\frac{47}{3}$	0	0	$-\frac{5}{3}$	0	$\frac{5}{3}$	$\frac{20}{3}$	(7)=(2)+5(10)
0	0	0	$\frac{1}{3}$	$\frac{11}{3}$	1	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{10}{3}$	(8)=(3) −2(10)
0	0	0	$-\frac{4}{3}$	$(\frac{13}{3})$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{11}{3}$	(9)=(4) −(10)
0	0	1	$\frac{1}{3}$	$-\frac{7}{3}$	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	(10)=(5)÷ 3

Step 6: As A hasn't yet been reduced to zero, we look for large positive coefficients of variables in the 1st row again, and so take z as the pivot column. Rows 7 and 10 can be ignored, when establishing the pivot row, due to their negative coefficients of z.

row 8: $\frac{(\frac{10}{3})}{(\frac{11}{3})} = \frac{10}{11}$; row 9: $\frac{(\frac{11}{3})}{(\frac{13}{3})} = \frac{11}{13} < \frac{10}{11}$, so the pivot row is row 9

(3) Maximise $5x - 2y + 4z$, subject to the following constraints:

$2x + y - z \leq 6$

$x - y + 2z \geq 5$

$3x + y - 7z \geq 4$

$x \geq 0, y \geq 0, z \geq 0$

Apply the Big M (Simplex) method, as far as establishing the pivot row for the 2nd time.

Solution

Step 1: (As for the 2 Stage method), create equations with either slack variables, or surplus and artifical variables, as appropriate

$P - 5x + 2y - 4z = 0$ (1)

$2x + y - z + s_1 = 6$ (2)

$x - y + 2z - s_2 + a_1 = 5$ (3)

$3x + y - 7z - s_3 + a_2 = 4$ (4)

Step 2: Modify the objective to maximising $P' = 5x - 2y + 4z - M(a_1 + a_2)$

$= 5x - 2y + 4z - M[(5 - x + y - 2z + s_2) + (4 - 3x - y + 7z + s_3)]$

$= (5 + 4M)x - 2y + (4 - 5M)z - Ms_2 - Ms_3 - 9M$

(where M is a large number)

Step 3: Represent the equations in a Simplex tableau:

P'	x	y	z	s_1	s_2	s_3	a_1	a_2	value	row
1	$-5 - 4M$	2	$-4 + 5M$	0	M	M	0	0	$-9M$	(1)
0	2	1	-1	1	0	0	0	0	6	(2)
0	1	-1	2	0	-1	0	1	0	5	(3)

0	(3)	1	-7	0	0	-1	0	1	4	(4)

Step 4: As we are maximising P', we look for large negative coefficients of variables in the 1st row (so that when the variable is maximised, it will increase P' as much as possible). Here we take x as the pivot column, and perform the ratio test to establish the pivot row.

row 2: $\frac{6}{2} = 3$; row 3: $\frac{5}{1} = 5$; row 4: $\frac{4}{3}$; so the pivot row is row 4 (indicated in the table above by the brackets - or circling if handwritten)

Step 5: Eliminate x from rows 1-3

P'	x	y	z	s_1	s_2	s_3	a_1	a_2	value	row
1	0	$\frac{4M+11}{3}$	$\frac{-13M-47}{3}$	0	M	$\frac{-M-5}{3}$	0	$\frac{5+4M}{3}$	$\frac{-11M+20}{3}$	(5)=(1)+ (5+4M)(8)
0	0	$\frac{1}{3}$	$\frac{11}{3}$	1	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{10}{3}$	(6)=(2) -2(8)
0	0	$-\frac{4}{3}$	$(\frac{20}{3})$	0	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{11}{3}$	(7)= (3) $-$ (8)
0	1	$\frac{1}{3}$	$-\frac{7}{3}$	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	(8)=(4)÷ 3

Step 6: As the value of P' still involves M, we look for large negative coefficients of variables in the 1st row again, and so take z as the pivot column. Row 8 can be ignored, when establishing the pivot row, due to its negative coefficient of z.

row 6: $\frac{(\frac{10}{3})}{(\frac{11}{3})} = \frac{10}{11}$; row 7: $\frac{(\frac{11}{3})}{(\frac{20}{3})} = \frac{11}{20} < \frac{10}{11}$, so the pivot row is row 7