Simplex Algorithm - Part 2 (16 pages; 3/11/16)

(1) The basic Simplex method (covered in Part 1) assumes that all the constraints are of the $\leq$ type (apart from $x \geq 0$, $y \geq 0$).

A constraint such as $3x + 2y - z \geq -2$ (as in Example 3 in Part 1) can just be rewritten as $-3x - 2y + z \leq 2$, but the constraint $3x + 2y - z \geq 2$ could not be dealt with in this way, as it would leave us with a negative value on the RHS.

There are two methods for dealing with such a problematic constraint: the 2-Stage Simplex method and the Big M (Simplex) method.

(2) 2-Stage Simplex Method

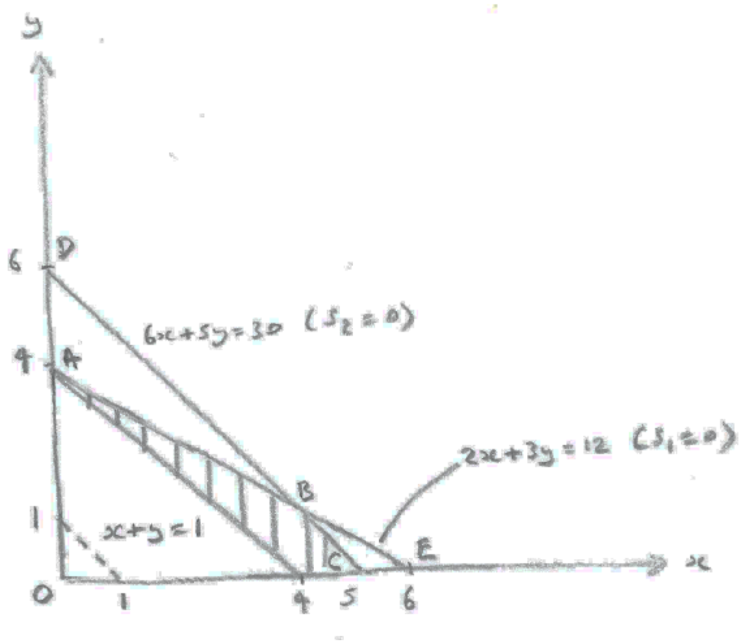
Example

Maximise $P = x + y$

subject to $2x + 3y \leq 12$

$6x + 5y \leq 30$

$x + y \geq 4$



Create slack variables as usual, but introduce an artificial variable for the $x + y \geq 4$ constraint.

$$P - x - y \qquad = 0 \quad (1)$$

$$2x + 3y + s_1 \qquad = 12 \quad (2)$$

$$6x + 5y + \quad s_2 = 30 \quad (3)$$

$$x + y - s_3 + a_1 = 4 \quad (4) \qquad\qquad (s_1,\ s_2, s_3,\ a_1 \geq 0)$$

a_1 is needed because $x + y - s_3 = 4$ doesn't allow $x = y = 0$, since $s_3 \geq 0$; but with the artificial variable we can now start with $x = y = s_3 = 0$ & $a_1 = 4$

Initial solution:

$x = y = s_3 = 0$; $s_1 = 12$, $s_2 = 30$, $a_1 = 4$, $P = 0$

The aim is to minimise a_1 , so that (if possible) the solution moves into the feasible region.

Create a new objective: minimise $A = a_1$

[It might seem a bit unnecessary to create another variable with the same value as a_1, but it enables the method to be extended easily to cases where there are two or more artificial variables - see later example.]

From (4), re-write $A = a_1$ as $A + x + y - s_3 = 4 \quad (5)$

Re-labelling the rows:

A	P	x	y	s_1	s_2	s_3	a_1		
1	0	1	1	0	0	-1	0	4	1
0	1	-1	-1	0	0	0	0	0	2
0	0	2	3	1	0	0	0	12	3
0	0	6	5	0	1	0	0	30	4
0	0	1	1	0	0	-1	1	4	5

Referring to the Simplex tableau above:

1st row: new objective (1st stage of method)

2nd row: original objective (2nd stage of method)

Each stage of the method involves applying the ordinary Simplex method.

To minimise A: the positive coefficients of x & y mean that x *or* y could be increased (alternatively maximise $-A$).

Choose x as the pivot column (eg) and apply the ratio test:

3rd row: $\frac{12}{2} = 6$, 4th row: $\frac{30}{6} = 5$, 5th row: $\frac{4}{1} = 4$

A	P	x	y	s_1	s_2	s_3	a_1		
1	0	0	0	0	0	0	−1	0	⑥ = ① − ⑩
0	1	0	0	0	0	−1	1	4	⑦ = ② + ⑩
0	0	0	1	1	0	2	−2	4	⑧ = ③ − 2 × ⑩
0	0	0	−1	0	1	6	−6	6	⑨ = ④ − 6 × ⑩
0	0	1	1	0	0	−1	1	4	⑩ = ⑤

As $A = 0$, the 1st stage has been successfully completed.

Now remove the 1st row, and the columns for A and a_1

(a_1 is a non-basic variable and is being set to 0)

Solution so far: $x = 4,\ y = 0,\ P = 4$

The 2nd stage is now to maximise P, as usual.

The remainder of the working is as follows:

P	x	y	s_1	s_2	s_3		
1	0	0	0	0	-1	4	7
0	0	1	1	0	2	4	8
0	0	-1	0	1	6	6	9
0	1	1	0	0	-1	4	10

Choose s_3 as the pivot column, and apply the ratio test:

8th row: $\frac{4}{2} = 2$, 9th row: $\frac{6}{6} = 1$, 10th row: n/a

P	x	y	s_1	s_2	s_3		
1	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	0	5	11 = 7 + 13
0	0	$\frac{4}{3}$	1	$-\frac{1}{3}$	0	2	12 = 8 − 2×13
0	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	1	1	13 = 9 ÷ 6
0	1	$\frac{5}{6}$	0	$\frac{1}{6}$	0	5	14 = 10 + 13

Choose y as the pivot column, and apply the ratio test:

12th row: $\frac{2}{\left(\frac{4}{3}\right)} = \frac{3}{2}$, 13th row: n/a , 14th row: $\frac{5}{\left(\frac{5}{6}\right)} = 6$

P	x	y	s_1	s_2	s_3		
1	0	0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{21}{4}$	(15) = (11) + $\frac{1}{6}$ × (16)
0	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	0	$\frac{3}{2}$	(16) = (12) × $\frac{3}{4}$
0	0	0	$\frac{1}{8}$	$\frac{1}{8}$	1	$\frac{5}{4}$	(17) = (13) + $\frac{1}{6}$ × (16)
0	1	0	$-\frac{5}{8}$	$\frac{3}{8}$	0	$\frac{15}{4}$	(18) = (14) − $\frac{5}{6}$ × (16)

The coefficients of s_1 & s_2 in (15) are both positive, so we have maximised P.

Solution: $x = \frac{15}{4}$, $y = \frac{3}{2}$, $P = \frac{21}{4}$ (B)

(3) The Big M (Simplex) Method (same example)

This starts off in the same way as the 2-Stage method, by creating the artifical variable.

$$P - x - y = 0 \quad (1)$$

$$2x + 3y + s_1 = 12 \quad (2)$$

$$6x + 5y + s_2 = 30 \quad (3)$$

$$x + y - s_3 + a_1 = 4 \quad (4)$$

We now modify the objective to:

maximise $P = x + y - Ma_1$, where M is a large number (eg 1000)

This ensures that minimising a_1 is given 1st priority, as the Ma_1 term has the biggest effect on P.

Re-write as $P = x + y - M(4 - x - y + s_3)$

giving $P - (1+M)x - (1+M)y + Ms_3 = -4M$

P	x	y	s_1	s_2	s_3	a_1		
1	$-(1+M)$	$-(1+M)$	0	0	M	0	$-4M$	①
0	2	3	1	0	0	0	12	②
0	6	5	0	1	0	0	30	③
0	1	1	0	0	-1	1	4	④

We now carry out the Simplex method as usual, and we should find that the RHS of the objective row becomes free of M.

Choose x as the pivot column (eg) and apply the ratio test:

2nd row: $\frac{12}{2} = 6$, 3rd row: $\frac{30}{6} = 5$, 4th row: $\frac{4}{1} = 4$ (as before)

P	x	y	s_1	s_2	s_3	a_1		
1	0	0	0	0	-1	$1+M$	4	⑤ = ① + $(1+M)\times$⑧
0	0	1	1	0	2	-2	4	⑥ = ② $- 2\times$⑧
0	0	-1	0	1	6	-6	6	⑦
0	1	1	0	0	-1	1	4	⑧ = ④

Once M only appears in the a_1 column, we can set a_1 to 0, and remove the a_1 column, arriving at the same tableau as at the end of the 1st stage of the 2-stage method (and then continue as before).

P	x	y	s_1	s_2	s_3		
1	0	0	0	0	−1	4	(5)
0	0	1	1	0	2	4	(6)
0	0	−1	0	1	6	6	(7)
0	1	1	0	0	−1	4	(8)

(4) Complications for the Simplex Method

The following is a summary of the various ways in which complications can arise.

(i) Objective function parallel to a constraint line (if two variables) or plane (if three).

As for the Linear Programming method, more than one solution is possible.

(ii) When applying the 2-Stage Simplex or Big M method, it may not be possible to reduce a_1 to 0; ie there may not be a solution to the problem.

(iii) Artificial variables may be needed for more than one constraint. In this case, let $A = a_1 + a_2 + \cdots$ for the 2-Stage Simplex, and have $-M(a_1 + a_2 + \cdots)$ in place of $-Ma_1$ for the

Big M method.

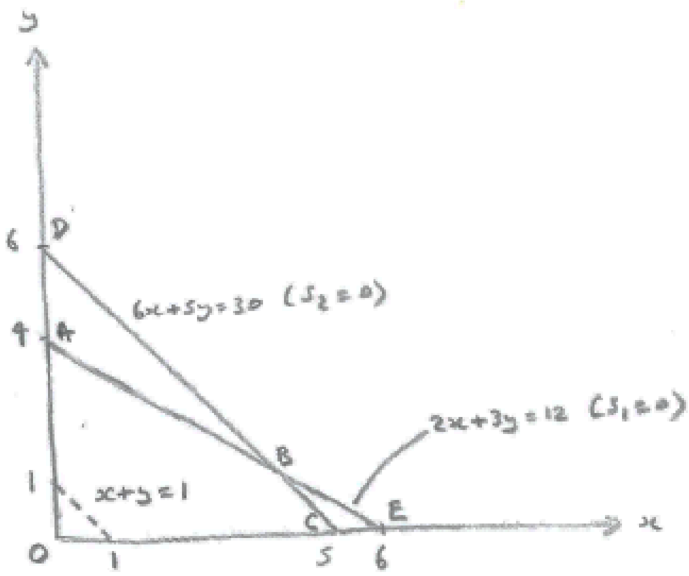
Example (2-Stage Simplex)

Maximise P $= x + y$

subject to $2x + 3y \geq 12$

$6x + 5y \leq 30$

$y \geq 5$



$P - x - y = 0 \quad (1)$

$2x + 3y - s_1 + a_1 = 12 \quad (2)$

$6x + 5y + s_2 = 30 \quad (3)$

$y - s_3 + a_2 = 5 \quad (4)$

Minimise $A = a_1 + a_2 = (12 - 2x - 3y + s_1) + (5 - y + s_3)$

$\Rightarrow A + 2x + 4y - s_1 - s_3 = 17$

min. max.

A	P	x	y	s_1	s_2	s_3	a_1	a_2		
1	0	2	4	-1	0	-1	0	0	17	(1)
0	1	-1	-1	0	0	0	0	0	0	(2)
0	0	2	3	-1	0	0	1	0	12	(3)
0	0	6	5	0	1	0	0	0	30	(4)
0	0	0	1	0	0	-1	0	1	5	(5)

1st stage

pivot column : y

ratio test : (2) n/a (3) $\frac{12}{3} = 4$ (4) $\frac{30}{5} = 6$ (5) $\frac{5}{1} = 5$

A	P	x	y	s_1	s_2	s_3	a_1	a_2		
1	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	-1	$-\frac{4}{3}$	0	1	(6) = (1) − 4×(8)
0	1	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0	4	(7) = (2) + (8)
0	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0	4	(8) = (3) ÷ 3
0	0	$2\frac{2}{3}$	0	$\frac{5}{3}$	1	0	$-\frac{5}{3}$	0	10	(9) = (4) − 5×(8)
0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	-1	$-\frac{1}{3}$	1	1	(10) = (5) − (8)

pivot column : s_1

ratio test : (7) n/a (8) n/a (9) $\frac{10}{\left(\frac{5}{3}\right)} = 6$ (10) $\frac{1}{\left(\frac{1}{3}\right)} = 3$

A	P	x	y	s_1	s_2	s_3	a_1	a_2		
1	0	0	0	0	0	0	−1	−1	0	11 = 6 − $\frac{1}{3}$ × 15
0	1	−1	0	0	0	−1	0	1	5	12 = 7 + $\frac{1}{3}$ × 15
0	0	0	1	0	0	−1	0	1	5	13 = 8 + $\frac{1}{3}$ × 15
0	0	6	0	0	1	5	0	−5	5	14 = 9 − $\frac{5}{3}$ × 15
0	0	−2	0	1	0	−3	−1	3	3	15 = 10 × 3

A has been minimised, with $a_1 = a_2 = 0$

max.

P	x	y	s_1	s_2	s_3		
1	−1	0	0	0	−1	5	12
0	0	1	0	0	−1	5	13
0	6	0	0	1	5	5	14
0	−2	0	1	0	−3	3	15

pivot column : x (s_3 is also possible)

ratio test : 14 is the only possible row

P	x	y	s_1	s_2	s_3		
1	0	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	$5\frac{5}{6}$	(16) = (12) + (18)
0	0	1	0	0	-1	5	(17) = (13)
0	1	0	0	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	(18) = (14) ÷ 6
0	0	0	1	$\frac{1}{3}$	$-\frac{4}{3}$	$4\frac{2}{3}$	(19) = (15) + 2 × (18)

pivot column : s_3

ratio test : (18) is the only possible row

P	x	y	s_1	s_2	s_3		
1	$\frac{1}{5}$	0	0	$\frac{1}{5}$	0	6	(20) = (16) + $\frac{1}{6}$ × (22)
0	$\frac{6}{5}$	1	0	$\frac{1}{5}$	0	6	(21) = (17) + (22)
0	$\frac{6}{5}$	0	0	$\frac{1}{5}$	1	1	(22) = (18) × $\frac{6}{5}$
0	$\frac{12}{5}$	0	1	$\frac{3}{5}$	0	6	(23) = (19) + $\frac{4}{3}$ × (22)

$\Rightarrow$ soln is : $x = 0$, $y = 6$ $\quad$ ($s_1 = 6$, $s_2 = 0$, $s_3 = 0$)

$P = 6$

Example (Big M):

Big M method

Maximise $P' = P - (a_1 + a_2)M$

$= x + y - (12 - 2x - 3y + s_1 + 5 - y + s_3)M$

$= (1+2M)x + (1+4M)y - Ms_1 - Ms_3 - 17M$

max.

P'	x	y	s_1	s_2	s_3	a_1	a_2		
1	$-(1+2M)$	$-(1+4M)$	M	0	M	0	0	$-17M$	1
0	2	3	-1	0	0	1	0	12	2
0	6	5	0	1	0	0	0	30	3
0	0	1	0	0	-1	0	1	5	4

pivot row : y

ratio test : 2 $\frac{12}{3} = 4$ 3 $\frac{30}{5} = 6$ 4 $\frac{5}{1} = 5$

P'	x	y	s_1	s_2	s_3	a_1	a_2		
1	$-\frac{1}{3}+\frac{2}{3}M$	0	$-\frac{1}{3}-\frac{1}{3}M$	0	M	$\frac{1}{3}+\frac{4}{3}M$	0	$4-M$	⑤ = ① + $(1+4M)$ × ⑥
0	$\frac{2}{3}$	1	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0	4	⑥ = ② ÷ 3
0	$2\frac{2}{3}$	0	$\frac{5}{3}$	1	0	$-\frac{5}{3}$	0	10	⑦ = ③ − 5 × ⑥
0	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	−1	$-\frac{1}{3}$	1	1	⑧ = ④ − ⑥

pivot row: s_1

ratio test: ⑥ n/a ⑦ $\frac{10}{(5/3)} = 6$ ⑧ $\frac{1}{(1/3)} = 3$

P'	x	y	s_1	s_2	s_3	a_1	a_2		
1	−1	0	0	0	−1	M	$1+M$	5	⑨ = ⑤ + $\frac{1}{3}(1+M)$ × ⑫
0	0	1	0	0	−1	0	1	5	⑩ = ⑥ + $\frac{1}{3}$ × ⑫
0	6	0	0	1	5	0	−5	5	⑪ = ⑦ − $\frac{5}{3}$ × ⑫
0	−2	0	1	0	−3	−1	3	3	⑫ = ⑧ × 3

then as per the 2-stage simplex method

(iv) Constraints that are equalities

Replace with two inequality constraints:

ie for $x + y = 4$: replace with $x + y \leq 4$ & $x + y \geq 4$

Example

Big M method

Maximise $x+y$, subject to $2x+y \leq 12$ and $x=4$

$2x + y + s_1 = 12$

$x - s_2 + a_1 = 4$

$x + s_3 = 4$

Modified objective: maximise $x+y - Ma_1$ (P)

$= x+y - M(4 - x + s_2)$

$= (1+M)x + y - Ms_2 - 4M$

P	x	y	s_1	s_2	s_3	a_1		
1	$-(1+M)$	-1	0	M	0	0	$-4M$	①
0	2	1	1	0	0	0	12	②
0	1	0	0	-1	0	1	4	③
0	1	0	0	0	1	0	4	④

pivot row : x

ratio test : ② $\frac{12}{2} = 6$ ③ $\underline{\frac{4}{1} = 4}$ ④ $\frac{4}{1} = 4$

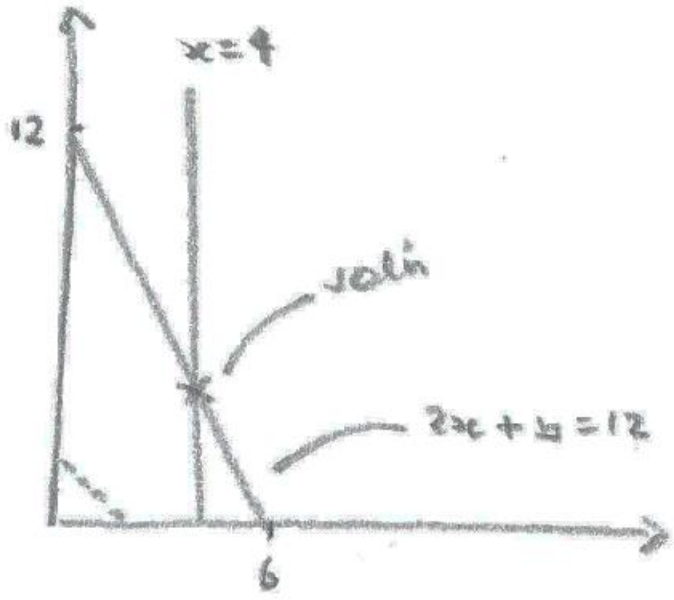
P	x	y	s_1	s_2	s_3	a_1		
1	0	-1	0	-1	0	$1+M$	4	⑤ = ① + $(1+M)$ × ⑦
0	0	1	1	2	0	-2	4	⑥ = ② − 2 × ⑦
0	1	0	0	-1	0	1	4	⑦ = ③
0	0	0	0	1	1	-1	0	⑧ = ④ − ⑦

pivot col : y ⇒ pivot row : ⑥

P	x	y	s_1	s_2	s_2		
1	0	0	1	1	0	8	⑨ = ⑤ + ⑩
0	0	1	1	2	0	4	⑩ = ⑥
0	1	0	0	-1	0	4	⑪ = ⑦
0	0	0	0	1	1	0	⑫ = ⑧

⇒ sol'n is : $x = 4$, $y = 4$ $(s_1 = 0, s_2 = 0, s_3 = 0)$

$P = 8$



(v) $x + y < 4$ (eg)

Use $x + y \leq 4$ instead, and reduce x *or* y slightly, if necessary.

(vi) Big M method: to minimise $P = x + y$

Modify to minimising $x + y + Ma_1$ (instead of maximising $x + y - Ma_1$)

(vii) If x (for example) can be negative, then replace x with $x_1 - x_2$, where $x_1, x_2 \geq 0$ (This allows x to be negative, if necessary.)