

Simplex Algorithm - Part 1 (15 pages; 9/1/24)

The Simplex algorithm provides an algebraic method of dealing with Linear Programming problems, suitable for use by a computer. It can deal with LP problems involving more than two variables that can't be represented on a 2D diagram.

The following examples should be read together. There is a summary of issues at the end.

Note: The Simplex tableau is only introduced in Example 3, to keep things simpler initially.

Example 1

maximise $P = x + y$

subject to $2x + 3y \leq 12$

$6x + 5y \leq 30$

$x \geq 0, y \geq 0$

The feasible region is indicated in Figure 1.

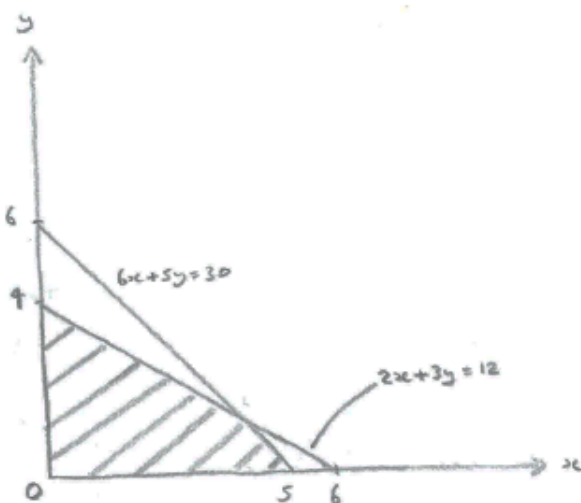


Figure 1

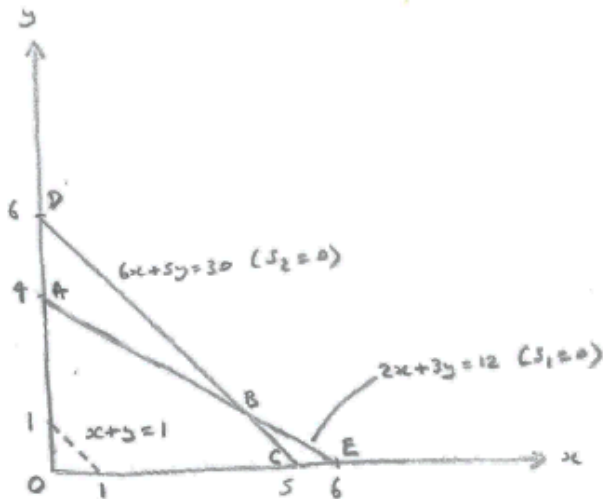


Figure 2

Referring to Figure 2, we know from LP that the maximum value of P will occur at one of the vertices of the feasible region. In this example it is B . This can be seen by examining lines parallel to $x + y = 1$: to obtain the maximum value of P , we want the line $P = x + y$ (with the same gradient as $x + y = 1$), which is furthest from the origin.

Step 1: Create equations, with slack variables:

$$P - x - y = 0 \quad (1)$$

$$2x + 3y + s_1 = 12 \quad (2)$$

$$6x + 5y + s_2 = 30 \quad (3) \quad (s_1, s_2 \geq 0)$$

(This form – where inequalities have been converted to equations - is sometimes referred to as the **augmented form**.)

Step 2: initial solution: $x = 0, y = 0, P = 0$ (corresponds to 0 in Figure 2)

Step 3: Referring to equation (1), P can be increased by increasing x or y . Choosing y for example:

$$(2), \text{ with } x = s_1 = 0 \Rightarrow y = \frac{12}{3} = 4 \quad [A]$$

$$(3), \text{ with } x = s_2 = 0 \Rightarrow y = \frac{30}{5} = 6 \quad [D] \text{ (but } s_1 < 0)$$

The largest possible value of y is therefore 4 (as it satisfies both (2) & (3)).

(This step is referred to as the '**ratio test**'.)

Equation (2) is described as the 'pivot row', with y being the 'pivot column'.

Step 4: Eliminate y from (1) & (3).

First of all, ensure that the coefficient of y in equation (2) is 1 (the reason for this will be clear shortly). Thus, equation (5) is created first.

$$P - \frac{x}{3} + \frac{s_1}{3} = 4 \quad (4)=(1)+(5)$$

$$\frac{2x}{3} + y + \frac{s_1}{3} = 4 \quad (5)=(2) \div 3$$

$$\frac{8x}{3} - \frac{5s_1}{3} + s_2 = 10 \quad (6)=(3)-5 \times (5)$$

Step 5: An improved solution is obtained with $y = 4$, and

$$x = 0, s_1 = 0, s_2 = 10, \text{ so that } P = 4$$

The reason for ensuring that the coefficient of y is 1 is so that the value of y can be read from the right-hand column (as can the value of P). It also simplifies the elimination of y by a computer (which can't just look for a clever way of doing it).

The improved solution corresponds to (A) in Figure 2.

Step 6: Referring to equation (4), P can be increased further by increasing x . Note that increasing s_1 is not an option, as P would have to be reduced. We therefore only consider variables in the 'objective row' (the row containing P) which have a negative coefficient.

$$(5) \text{ with } y = 0 \ \& \ s_1 = 0 \Rightarrow x = \frac{4}{\left(\frac{2}{3}\right)} = 6 \quad (\text{E})$$

$$(6) \text{ with } s_1 = s_2 = 0 \Rightarrow x = \frac{10}{\left(\frac{8}{3}\right)} = \frac{15}{4} \quad (\text{B})$$

As $\frac{15}{4}$ is the smaller of the two, choose equation (6) as the pivot row.

Step 7: Create equation (9), and then eliminate x from eq'ns (4) & (5):

$$P + \quad \frac{s_1}{8} + \frac{s_2}{8} = \frac{21}{4} \quad (7) = (4) + \frac{1}{3} \times (9)$$

$$y + \frac{3s_1}{4} - \frac{s_2}{4} = \frac{3}{2} \quad (8) = (5) - \frac{2}{3} \times (9)$$

$$x \quad -\frac{5s_1}{8} + \frac{3s_2}{8} = \frac{15}{4} \quad (9) = (6) \times \frac{3}{8}$$

Step 8: An improved solution is $x = \frac{15}{4}$, $y = \frac{3}{2}$, $P = \frac{21}{4}$ (B)

(with $s_1 = s_2 = 0$)

Step 9: As the coefficients of s_1 & s_2 are both positive, there is no further scope for increasing P.

Example 2

Maximise $P = 2x + 3y$

subject to the following constraints:

$$x + 2y \leq 12$$

$$3x + y \leq 15$$

$$y \leq 5$$

$$x \geq 0, y \geq 0$$

Step 1: Create equations, with slack variables:

$$P - 2x - 3y = 0 \quad (1)$$

$$x + 2y + s_1 = 12 \quad (2)$$

$$3x + y + s_2 = 15 \quad (3)$$

$$y + s_3 = 5 \quad (4)$$

This can be represented graphically as in Figure 3.

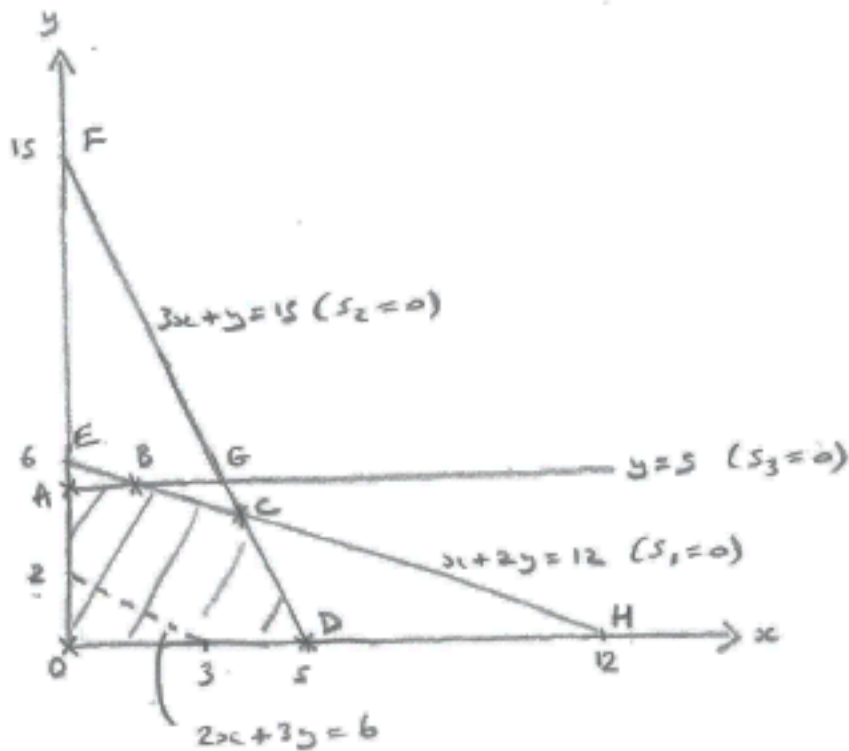


Figure 3

Step 2: initial solution: $x = 0, y = 0, P = 0$ (corresponds to O in Figure 3)

Step 3: Referring to equation (1), P can be increased by increasing x or y , as the coefficients of both x and y are negative. Because the coefficient of y is greater in magnitude, the greatest increase in P is likely to arise from maximising y (all else being equal). Hence make y the pivot column. (This isn't essential: we could have used x instead, and arrived at the same result in the end, by a different route.)

Then (2) with $x = s_1 = 0 \Rightarrow y = \frac{12}{2} = 6$ [E]

(3) with $x = s_2 = 0 \Rightarrow y = 15$ [F]

(4) with $s_3 = 0 \Rightarrow y = 5$ [A]

$y = 5$ is the only one that satisfies (2), (3) & (4), and is therefore the largest value of y that we can have (eg $y = 6$, at E, would make $s_3 < 0$ in (4))

So use (4) to eliminate y from the other equations. (By doing this we are not infringing any of the constraints: use of (2) instead, for example, could leave us with $s_3 < 0$ at the end.)

Step 4: Eliminate y from (1), (2) & (3)

$$P - 2x \qquad \qquad + 3s_3 = 15 \quad (5)=(1)+3\times(8)$$

$$x \quad + s_1 \quad - 2s_3 = 2 \quad (6)=(2)-2\times(8)$$

$$3x \qquad \quad + s_2 - s_3 = 10 \quad (7)=(3)-(8)$$

$$y \qquad \qquad + s_3 = 5 \quad (8)=(4)$$

Step 5: An improved solution is obtained with $y = 5$, and $x = s_1 = s_2 = 0$, so that $P = 15$. (The values of y and P can be read from the right-hand side of equations (5) & (8). Note that y and P are the variables which only appear in one row, with the other variables being set equal to zero. As y and P only appear in one row, their values don't affect other rows.)

The improved solution corresponds to (A) in Figure 3.

Step 6: Referring to equation (5), P can be increased further by increasing x (but not s_3 , as its coefficient is positive). Had the coefficient of x been positive at this stage, we would have stopped, with $x = 0$.

In (6), $s_1 = s_3 = 0 \Rightarrow x = 2$ [B]

In (7), $s_2 = s_3 = 0 \Rightarrow x = \frac{10}{3}$ [G]

We choose the lower of the two: $x = 2$, and select (6) as the pivot row.

Step 7: Eliminating x from eq'ns (5) & (7):

$$P \quad + 2s_1 \quad - s_3 = 19 \quad (9)=(5)+2\times(10)$$

$$x \quad + s_1 \quad - 2s_3 = 2 \quad (10)=(6)$$

$$\quad - 3s_1 + s_2 + 5s_3 = 4 \quad (11)=(7)-3\times(10)$$

$$y \quad + s_3 = 5 \quad (12)=(8)$$

Step 8: An improved solution is $x = 2$, $y = 5$, $P = 19$ (B)

(with $s_1 = s_2 = s_3 = 0$)

Step 9: The coefficient of s_3 is now negative, so we can increase s_3 .

Note that (in general) we may end up reducing x or y (as we move to another vertex on the feasible region).

In (10) the coefficient of s_3 is -2, and so (10) isn't considered (as $x = s_1 = 0 \Rightarrow s_3 < 0$: $x = s_1 = 0 \Rightarrow$ [E])

In (11) $s_1 = s_2 = 0 \Rightarrow s_3 = \frac{4}{5}$ [C]

In (12) $(s_1) = y = 0 \Rightarrow s_3 = 5$ [H]

As $\frac{4}{5} < 5$, (11) is selected as the pivot row.

Step 10: Adjusting (11) so that the coefficient of s_3 is 1, and then eliminating s_3 from eq'ns (13), (14) & (16):

$$\begin{array}{rcll}
 P & +\frac{7}{5}s_1 + \frac{1}{5}s_2 & = 19\frac{4}{5} & (13)=(9)+(15) \\
 x & -\frac{1}{5}s_1 + \frac{2}{5}s_2 & = 3\frac{3}{5} & (14)=(10)+2\times(15) \\
 & -\frac{3}{5}s_1 + \frac{1}{5}s_2 + s_3 & = \frac{4}{5} & (15)=(11)\div 5 \\
 y & +\frac{3}{5}s_1 - \frac{1}{5}s_2 & = 4\frac{1}{5} & (16)=(12)-(15)
 \end{array}$$

Step 11: As the coefficients of s_1 & s_2 are both positive, there is no further scope for increasing P .

The solution is $s_1 = s_2 = 0$; $P = 19\frac{4}{5}$

$$x = 3\frac{3}{5}, s_3 = \frac{4}{5}, y = 4\frac{1}{5}$$

corresponding to C

Example 3

Minimise $2x + y - 3z$

subject to the following constraints:

$$x - 4y + z \leq 4$$

$$3x + 2y - z \geq -2$$

$$x \geq 0, y \geq 0, z \geq 0$$

Note that $(0,0,0)$ is a feasible solution, which can be taken as the starting point.

Step 1

Rewrite as: Maximise $P = -2x - y + 3z$

subject to $x - 4y + z \leq 4$ and $-3x - 2y + z \leq 2$

ensuring that all right-hand values are non-negative (excluding the objective row) [by doing this, we need only consider positive coefficients in the pivot columns; negative coefficients won't give a positive value for the variable being maximised]

Step 2: Create equations, with slack variables:

$$P + 2x + y - 3z = 0 \quad (1)$$

$$x - 4y + z + s_1 = 4 \quad (2)$$

$$-3x - 2y + z + s_2 = 2 \quad (3)$$

Step 3: Represent the equations in a Simplex tableau

P	x	y	z	s_1	s_2		equation
1	2	1	-3	0	0	0	①
0	1	-4	1	1	0	4	②
0	-3	-2	1	0	1	2	③

Step 4

Choose z as the pivot column (as it is the only variable in the objective row with a negative coefficient).

$$(2), \text{ with } x = y = s_1 = 0 \Rightarrow z = 4$$

$$(3), \text{ with } x = y = s_2 = 0 \Rightarrow z = 2$$

As $2 < 4$, (3) is the pivot row (indicated by circling the coefficient of z in equation 3 in the table below).

Step 5: Eliminate z from equations 1 & 2:

P	x	y	z	s_1	s_2		Equation
1	2	1	-3	0	0	0	①
0	1	-4	1	1	0	4	②
0	-3	-2	①	0	1	2	③
1	-7	-5	0	0	3	6	④ = ① + 3 × ③
0	4	-2	0	1	-1	2	⑤ = ② - ③
0	-3	-2	1	0	1	2	⑥ = ③

Steps 6, 7 & 8

x now has the largest negative coefficient in the objective row (equation 4), so this is the next pivot column.

As the only positive coefficient in the x column occurs in equation 5, this is the pivot row.

As the coefficient of x is 4, equation 5 needs to be divided by 4, before x is eliminated from the other equations.

P	x	y	z	s_1	s_2	Equation	
1	2	1	-3	0	0	0	①
0	1	-4	1	1	0	4	②
0	-3	-2	①	0	1	2	③
1	-7	-5	0	0	3	6	④ = ① + 3 × ③
0	④	-2	0	1	-1	2	⑤ = ② - ③
0	-3	-2	1	0	1	2	⑥ = ③
1	0	$-8\frac{1}{2}$	0	$\frac{3}{4}$	$\frac{1}{4}$	$9\frac{1}{2}$	⑦ = ④ + 7 × ⑤
0	1	$-\frac{1}{2}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	⑧ = ⑤ + ④
0	0	$-3\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{1}{4}$	$3\frac{1}{2}$	⑨ = ⑥ + 3 × ⑤

Step 9

There are no further variables that have negative coefficients in the objective row and positive coefficients elsewhere in their columns, so we have reached the end of the process.

The variables with a single 1 (and otherwise 0s) in their columns are termed the '**basic**' variables, and their values can be read off the right-hand column, whilst the other ('**non-basic**') variables are set to zero.

(The basic variables can be set equal to the right-hand value without affecting the other rows, as they only appear in one row.)

Solution: $x = 0.5$, $y = 0$, $z = 3.5$, $s_1 = 0$, $s_2 = 0$

Minimised value of $2x + y - 3z$ is 9.5

Step 10: Check

$$x - 4y + z = 4(\leq 4)$$

$$3x + 2y - z = -2(\geq -2)$$

Note: Minimisation problems, or problems involving \geq (or $=$) constraints, or problems where one of the variables could be negative are sometimes described as being in **non-standard form**. (See Part 2.)

Summary

(It is assumed that $x, y, z \geq 0$)

(1) Ensure that $x = y (= z) = 0$ is a feasible solution (in order for the method to be applied)

(2) To minimise the objective function, set P equal to $-1 \times$ the function

(3) Ensure that the right-hand values are all ≥ 0 (apart from the objective row). By doing this, we need only consider positive coefficients in the pivot columns (negative coefficients won't give a positive value for the variable being maximised). Also, see the Further Note below.

(4) Create equations involving non negative slack variables. Represent them in the Simplex tableau.

(5) In order to increase P, choose a variable with a negative coefficient in the objective row (either x, y or z , or one of the slack variables). Ideally choose the variable with the largest absolute coefficient (say, x in this example).

(6) For each row (other than the objective row), examine the effect of setting other variables equal to zero; choose the lowest value of x that results (ie the value that will satisfy all rows). This is the maximum value of x , which enables P to be increased as much as possible.

To do this, we carry out the ratio test (dividing the right-hand value by the coefficient of x in the pivot column). This establishes the pivot row.

Note: Only positive coefficients of x need be considered (for the reason mentioned in (2)).

(7) Divide the pivot row by the coefficient of x [this makes it easier to read off the solution in (7)]

Then eliminate x from the other rows.

(8) Repeat for other variables in the objective row (apart from P), and stop when the coefficients of the variables in the objective row are all positive, or if there is a negative value, but no positive values in the column. These variables are then set equal to 0 in the solution (in order to maximise P).

The basic variables (the ones that aren't set equal to 0) will have a single 1 in their columns (and 0s otherwise).

The solution is then obtained by setting each of the basic variables to the right-hand value (in the row where their 1 appears).

The maximised value of P is the right-hand value in the objective row.

Further Note

Why can rows with negative coefficients in the pivot column be ignored?

Consider the following example:

$$P - x - y = 0 \quad (1)$$

$$-2x + 3y + s_1 = 12 \quad (2)$$

$$6x + 5y + s_2 = 30 \quad (3)$$

Applying the ratio test to maximise x in $6x + 5y + s_2 = 30$
 (3) gives $y = s_2 = 0; x = \frac{30}{6}$

Because of the negative coefficient of x , row (2) places no upper limit on the value of x ; eg we could have $x = 100, y = 0, s_1 = 212$

So the only row that requires the ratio test for x to be carried out is row (3).

Part 2 of this note covers the various ways of dealing with possible complications.