Series - Results (3 pages; 1/7/26)

References:

(A) "Euler – The master of us all" (William Dunham; The Mathematical Association of America)

Binomial Series and Coefficients

(1) $(1-x)^{-1} = \sum_{r=0}^{\infty} x^r$ (Geometric series)

(2) $(1-x)^{-\frac{1}{2}} = \sum_{r=0}^{\infty} \frac{1}{2^{2r}} \binom{2r}{r} x^r$ (see STEP 2024/P2/Q6)

(3) $\sum_{r=0}^{n} \binom{n}{r} = 2^n$ [from $(1+1)^n$]

(4) $\sum_{r=0}^{n} r \binom{n}{r} = n2^{n-1}$

$\sum_{r=1}^{\infty} \frac{1}{r^p}$

(1) $\sum_{r=1}^{\infty} \frac{1}{r} = \infty$ ('harmonic' series)

(2) $\sum_{prime\ p} (\frac{1}{p}) = \infty$

(3) $\sum_{perfect\ square\ n} (\frac{1}{n}) = \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$ (Euler [1707-1783] - see Reference (A) for a proof)

(4) Euler's constant, $\gamma = \lim_{n\to\infty}[(\sum_{r=1}^{n}\frac{1}{r}) - \ln(n)] = 0.57722\ (5sf)$

This has been shown to be equal to $-\int_0^\infty e^{-x} lnx\, dx$,

and also $\lim_{x\to 1^+}\sum_{r=1}^{\infty}(\frac{1}{r^x} - \frac{1}{x^r})$.

But it has never been proven to be irrational (though it is definitely expected to be transcendental – ie an irrational number that isn't the root of an algebraic equation with rational coefficients; other examples being π and e).

(5) $\sum_{r=1}^{\infty}\frac{1}{r} = \prod_{primes\ p}(\frac{1}{1-\frac{1}{p}})$ (where Π denotes a product)[Euler]

[(4) and (5) show that there is a connection between prime numbers, the harmonic series, and the natural logarithm; the Prime Number theorem (initially a conjecture by Gauss [30/4/1777-23/2/1855]) states that $\frac{\pi(x)}{x} \to \frac{1}{lnx}$ as $x \to \infty$, where $\pi(x)$ is the number of primes less than or equal to x]

(6) $\sum_{r=1}^{\infty}\frac{1}{r^k} = \prod_{primes\ p}(\frac{1}{1-\frac{1}{p^k}})$ for $k \geq 1$ (Kronecker [1823-1891])

(7) As $\sum_{r=1}^{\infty}\frac{1}{r^2}$ is finite , $\sum_{r=1}^{\infty}\frac{1}{r^k}$ must be finite for $k \geq 2$ (as $\frac{1}{r^k} \leq \frac{1}{r^2}$ for each r).

In fact it has been shown that $\sum_{r=2}^{\infty}\frac{1}{r^k} \leq \frac{1}{k-1}$ (for $k \geq 2$)

(8) $\sum_{r=1}^{\infty} \frac{1}{r^3}$ is still undetermined (it is approximately 1.2021 and known as Apéry's constant), but others have been found (generally by Euler); for example: $\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90}$ and $\sum_{r=1}^{\infty} \frac{1}{r^6} = \frac{\pi^6}{945}$

Miscellaneous

(1) $\sum_{r=0}^{\infty} \frac{1}{2r+1}(-1)^r = \frac{\pi}{4}$ and $\sum_{r=0}^{\infty} \frac{1}{(2r+1)^3}(-1)^r = \frac{\pi^3}{32}$

(2) $\sum_{r=1}^{\infty} \frac{r^2}{2^r} = 6$ and $\sum_{r=1}^{\infty} \frac{r^3}{2^r} = 26$

(3) $\sum_{r=1}^{\infty} \frac{1}{r^r} \approx 1.2913$ $[\int_0^1 \frac{1}{x^x}\, dx\ ]$