Series: Further Exercises (with Sol'ns)

(16 pages; 27/3/26)

By considering $\sum_{r=1}^{n}(r+1)^3 - \sum_{r=1}^{n} r^3$ (*), show that

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ [assuming that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$]

By considering $\sum_{r=1}^{n}(r+1)^3 - \sum_{r=1}^{n} r^3$ (*), show that

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ [assuming that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$]

Solution

$(*) = 3(\sum_{r=1}^{n} r^2) + 3(\sum_{r=1}^{n} r) + n$

Also, $(*) = (n+1)^3 - 1$

Hence $3(\sum_{r=1}^{n} r^2) = (n+1)^3 - 1 - \frac{3}{2}n(n+1) - n$

$\Rightarrow 6(\sum_{r=1}^{n} r^2) = (n+1)\{2(n^2+2n+1) - 2 - 3n\}$

$= (n+1)(2n^2+n)$

$= n(n+1)(2n+1)$

$\Rightarrow \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

By considering $\sum_{r=1}^{n}(r+1)^4$, show that

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ [The results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ can be assumed.]

By considering $\sum_{r=1}^{n}(r+1)^4$, show that

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Solution

$$\sum_{r=1}^{n}(r+1)^4 = \left(\sum_{r=1}^{n} r^4\right) + 4\left(\sum_{r=1}^{n} r^3\right) + 6\left(\sum_{r=1}^{n} r^2\right) + 4\left(\sum_{r=1}^{n} r\right) + n$$

Let $R = r + 1$, so that LHS $= \sum_{R=2}^{n+1} R^4 = \left(\sum_{R=1}^{n} R^4\right) + (n+1)^4 - 1$

Thus $\left(\sum_{R=1}^{n} R^4\right) + (n+1)^4 - 1 = \left(\sum_{r=1}^{n} r^4\right) + 4\left(\sum_{r=1}^{n} r^3\right) + 6\left(\sum_{r=1}^{n} r^2\right) + 4\left(\sum_{r=1}^{n} r\right) + n$

and so $(n+1)^4 - 1 = 4\left(\sum_{r=1}^{n} r^3\right) + 6\left(\sum_{r=1}^{n} r^2\right) + 4\left(\sum_{r=1}^{n} r\right) + n$

Hence $4\left(\sum_{r=1}^{n} r^3\right) = n^4 + 4n^3 + 6n^2 + 4n - n(n+1)(2n+1) - 2n(n+1) - n$

$= n^4 + 4n^3 + 6n^2 + 4n - (2n^3 + 3n^2 + n) - 2n^2 - 2n - n$

$= n^4 + 2n^3 + n^2 = n^2(n^2 + 2n + 1) = n^2(n+1)^2$

and $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Starting from $\sum_{r=1}^{n}(r+1)^2 = \left(\sum_{r=1}^{n} r^2\right) + 2\left(\sum_{r=1}^{n} r\right) + n$,

make the substitution $R = r + 1$,

to prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Starting from $\sum_{r=1}^{n}(r+1)^2 = (\sum_{r=1}^{n} r^2) + 2(\sum_{r=1}^{n} r) + n$,

make the substitution $R = r + 1$,

to prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Solution

Let $R = r + 1$, so that

LHS $= \sum_{R=2}^{n+1} R^2 = (\sum_{R=1}^{n} R^2) + (n+1)^2 - 1$

Thus $(\sum_{R=1}^{n} R^2) + (n+1)^2 - 1 = (\sum_{r=1}^{n} r^2) + (2\sum_{r=1}^{n} r) + n$,

As $(\sum_{R=1}^{n} R^2)$ and $(\sum_{r=1}^{n} r^2)$ are equal, it follows that

$2\sum_{r=1}^{n} r = (n+1)^2 - 1 - n = n^2 + n = n(n+1)$

and so $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Note: This method is a variant on the "Method of Differences".

Given that $\frac{2r+1}{r^2(r+1)^2} = \frac{1}{r^2} - \frac{1}{(r+1)^2}$, use the method of differences to find $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$

Given that $\frac{2r+1}{r^2(r+1)^2} = \frac{1}{r^2} - \frac{1}{(r+1)^2}$, use the method of differences to find $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$

Solution

$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n} \frac{1}{r^2} - \frac{1}{(r+1)^2}$

$= (\frac{1}{1^2} + [\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2} + \frac{1}{n^2}])$

$-([\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots + \frac{1}{n^2}] + \frac{1}{(n+1)^2})$

As the terms in square brackets cancel:

$= \frac{1}{1^2} - \frac{1}{(n+1)^2} = \frac{(n+1)^2-1}{(n+1)^2}$

$= \frac{n^2+2n}{(n+1)^2}$

$= \frac{n(n+2)}{(n+1)^2}$

Given that $\frac{1}{2r} - \frac{1}{2(r+2)} = \frac{1}{r(r+2)}$, use the method of differences to find $\sum_{r=1}^{n} \frac{1}{r(r+2)}$

Given that $\frac{1}{2r}-\frac{1}{2(r+2)}=\frac{1}{r(r+2)}$, use the method of differences to find $\sum_{r=1}^{n}\frac{1}{r(r+2)}$

Solution

$\sum_{r=1}^{n}\frac{1}{r(r+2)}=(\sum_{r=1}^{n}\frac{1}{2r})-\left(\sum_{r=1}^{n}\frac{1}{2(r+2)}\right)$

$=\left(\frac{1}{2}+\frac{1}{4}+[\frac{1}{6}+\frac{1}{8}+\cdots+\frac{1}{2(n-2)}+\frac{1}{2(n-1)}+\frac{1}{2n}]\right)$

$-\left([\frac{1}{6}+\frac{1}{8}\ldots+\frac{1}{2n}]+\frac{1}{(2n+2)}+\frac{1}{(2n+4)}\right)$

As the terms in square brackets cancel:

$=\frac{1}{2}+\frac{1}{4}-\frac{1}{(2n+2)}-\frac{1}{(2n+4)}$

$=\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$

[For exam purposes, it may not be necessary to write this as a single fraction.]

$=\frac{1}{4(n+1)(n+2)}\{3(n+1)(n+2)-2(n+2)-2(n+1)\}$

$=\frac{1}{4(n+1)(n+2)}\{3n^2+9n+6-2n-4-2n-2\}$

$=\frac{1}{4(n+1)(n+2)}\{3n^2+5n\}$

$=\frac{n(3n+5)}{4(n+1)(n+2)}$

[This can be checked by substituting $n=1$.]

Given that $\frac{1}{2r-1} - \frac{2}{2r+1} + \frac{1}{2r+3} = \frac{8}{(2r-1)(2r+1)(2r+3)}$, use the method of differences to show that

$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)(2r+3)} = \frac{n(n+2)}{3(2n+1)(2n+3)}$

Given that $\frac{1}{2r-1}-\frac{2}{2r+1}+\frac{1}{2r+3}=\frac{8}{(2r-1)(2r+1)(2r+3)}$, use the method of differences to show that

$$\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)(2r+3)}=\frac{n(n+2)}{3(2n+1)(2n+3)}$$

Solution

$$\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)(2r+3)}=\frac{1}{8}\{\sum_{r=1}^{n}\frac{1}{2r-1}-\sum_{r=1}^{n}\frac{2}{2r+1}+\sum_{r=1}^{n}\frac{1}{2r+3}\}$$

$$=\frac{1}{8}\{\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2n-5}+\frac{1}{2n-3}+\frac{1}{2n-1}$$

$$-2(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots+\frac{1}{2n-3}+\frac{1}{2n-1}+\frac{1}{2n+1})$$

$$+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\cdots+\frac{1}{2n-1}+\frac{1}{2n+1}+\frac{1}{2n+3}\}$$

As the highlighted items cancel:

$$=\frac{1}{8}\{1-\frac{1}{3}-\frac{1}{2n+1}+\frac{1}{2n+3}\}$$

$$=\frac{2(2n+1)(2n+3)-3(2n+3)+3(2n+1)}{24(2n+1)(2n+3)}$$

$$=\frac{8n^2+16n}{24(2n+1)(2n+3)}$$

$$=\frac{n(n+2)}{3(2n+1)(2n+3)}$$

Given that $\frac{1}{r+1}-\frac{2}{r+2}+\frac{1}{r+3}=\frac{2}{(r+1)(r+2)(r+3)}$, use the method of differences to show that $\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)(r+3)}=\frac{n(n+5)}{12(n+2)(n+3)}$

Given that $\frac{1}{r+1}-\frac{2}{r+2}+\frac{1}{r+3}=\frac{2}{(r+1)(r+2)(r+3)}$, use the method of differences to show that $\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)(r+3)}=\frac{n(n+5)}{12(n+2)(n+3)}$

Solution

$$\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)(r+3)}=\frac{1}{2}\{\sum_{r=1}^{n}\frac{1}{r+1}-\sum_{r=1}^{n}\frac{2}{r+2}+\sum_{r=1}^{n}\frac{1}{r+3}\}$$

$$=\frac{1}{2}\{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}$$

$$-2(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2})$$

$$+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3})\}$$

As the highlighted items cancel:

$$=\frac{1}{2}\{\frac{1}{2}-\frac{1}{3}-\frac{1}{n+2}+\frac{1}{n+3}\}$$

$$=\frac{(n+2)(n+3)-6(n+3)+6(n+2)}{12(n+2)(n+3)}$$

$$=\frac{n^2+5n}{12(n+2)(n+3)}$$

$$=\frac{n(n+5)}{12(n+2)(n+3)}$$

Show that $\sum_{r=1}^{3k} r\tan(60r)^{\circ} = -k\sqrt{3}$

Show that $\sum_{r=1}^{3k} rtan(60r)^{\circ} = -k\sqrt{3}$

Solution

$\sum_{r=1}^{3k} rtan(60r)^{\circ}$

Consider

$\tan 60\,^{\circ} + 2tan120^{\circ} + 3tan180^{\circ} = \sqrt{3} + 2\tan\,(120 - 180)^{\circ}$

$+3tan0^{\circ}$

$= \sqrt{3} + 2\tan(-60^{\circ}) + 0$

$= \sqrt{3} - 2(\sqrt{3}) = -\sqrt{3}$

Also, $4\tan 240\,^{\circ} + 5tan300^{\circ} + 6tan360^{\circ}$

$= 4\tan(240 - 180)\,^{\circ} + 5\tan(300 - 360)\,^{\circ} + 0$

$= 4\tan(60^{\circ}) + 5\tan\,(-60^{\circ})$

$= 4\sqrt{3} - 5\sqrt{3} = -\sqrt{3}$

In general, $(3\text{p+}1)tan(60(3p + 1))^{\circ}$

$+(3\text{p+}2)tan(60(3p + 2))^{\circ}$

$+(3\text{p+}3)tan(60(3p + 3))^{\circ}$

$= (3p + 1)tan60^{\circ} + (3p + 2)\tan(120^{\circ})$

$+(3p + 3)\tan\,[180(p + 1)]^{\circ}$

$= (3p + 1)\sqrt{3} + (3p + 2)\tan(-60^{\circ}) + 0$

$= (3p + 1)\sqrt{3} - (3p + 2)\sqrt{3} = -\sqrt{3}$

So, as $\sum_{r=1}^{3k} rtan(60r)^{\circ}$ has k groups of expressions of the previous form, it equals $= -k\sqrt{3}$