

Series: Further Exercises (2 pages; 27/3/26)

(1) By considering $\sum_{r=1}^n (r+1)^3 - \sum_{r=1}^n r^3$ (*), show that

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1) \text{ [assuming that } \sum_{r=1}^n r = \frac{1}{2}n(n+1)]$$

(2) $\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$ [The results for $\sum_{r=1}^n r$ and $\sum_{r=1}^n r^2$ can be assumed.]

(3) By considering $\sum_{r=1}^n (r+1)^4$, show that

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

(4) Starting from $\sum_{r=1}^n (r+1)^2 = (\sum_{r=1}^n r^2) + 2(\sum_{r=1}^n r) + n$,

make the substitution $R = r + 1$,

to prove that $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$

(5) Given that $\frac{2r+1}{r^2(r+1)^2} = \frac{1}{r^2} - \frac{1}{(r+1)^2}$, use the method of differences to find $\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2}$

(6) Given that $\frac{1}{2r} - \frac{1}{2(r+2)} = \frac{1}{r(r+2)}$, use the method of differences to find $\sum_{r=1}^n \frac{1}{r(r+2)}$

(7) Given that $\frac{1}{2r-1} - \frac{2}{2r+1} + \frac{1}{2r+3} = \frac{8}{(2r-1)(2r+1)(2r+3)}$, use the method of differences to show that

$$\sum_{r=1}^n \frac{1}{(2r-1)(2r+1)(2r+3)} = \frac{n(n+2)}{3(2n+1)(2n+3)}$$

(8) Given that $\frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3} = \frac{2}{(r+1)(r+2)(r+3)}$, use the method of differences to show that $\sum_{r=1}^n \frac{1}{(r+1)(r+2)(r+3)} = \frac{n(n+5)}{12(n+2)(n+3)}$

(9) Show that $\sum_{r=1}^{3k} r \tan(60r)^\circ = -k\sqrt{3}$