

Series – Ideas & Exercises (3 pages ; 27/12/24)

$$\sum_{r=1}^n r = 1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n + 1)$$

[Informal proof: The average size of the terms being added is $\frac{1}{2}(1 + n)$, and there are n terms.]

$$\sum_{r=1}^n r^2 = \frac{1}{6} n(n + 1)(2n + 1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4} n^2(n + 1)^2$$

Arithmetic sequences and series

[Note: ‘series’ $\Rightarrow$ the terms of the sequence are added.]

Inductive/iterative/recurrence formula

$$a_{r+1} = a_r + d ; a_1 = a$$

Deductive/direct formula

$$a_r = a + (r - 1)d = dr + (a - d) = dr + a_0$$

[compare with the straight line $y = mx + c$, with $x = r, y = a_r$,
 $m = d$ & $c = a_0 = a_1 - d$; noting that a_0 is the hypothetical ‘0th’
 term of the sequence.]

Method of Differences

Example: To find $\sum_{r=1}^n \frac{1}{(r+1)(r+2)(r+3)}$

Step 1: Writing $\frac{1}{(r+1)(r+2)(r+3)} = \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3}$ (method of Partial Fractions),

$$\text{RHS} = \frac{A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2)}{(r+1)(r+2)(r+3)}, \text{ so that}$$

$$1 = A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2) \quad (*)$$

Setting $r = -2$, $1 = -B$; $B = -1$

Setting $r = -3$, $1 = 2C$; $C = \frac{1}{2}$

Setting $r = -1$, $1 = 2A$; $A = \frac{1}{2}$

[As a check, equating the coefficients of r^2 (as $(*)$ has to hold for all r), $0 = A + B + C$]

$$\text{Thus } \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2(r+1)} - \frac{1}{r+2} + \frac{1}{2(r+3)}$$

$$\text{Step 2: } \sum_{r=1}^n \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2} \sum_{r=1}^n \left\{ \frac{1}{(r+1)} - \frac{2}{(r+2)} + \frac{1}{(r+3)} \right\} = \frac{1}{2} S,$$

$$\begin{aligned} \text{where } S = & \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n} + \frac{1}{n+1} \right) \\ & - 2 \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} \right) \\ & + \left(\frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} \right) \end{aligned}$$

The bold terms cancel, so that

$$\begin{aligned} \sum_{r=1}^n \frac{1}{(r+1)(r+2)(r+3)} &= \frac{1}{2} \left\{ \frac{5}{6} - \frac{2}{3} - \frac{1}{n+2} + \frac{1}{n+3} \right\} \\ &= \frac{1}{12} - \frac{1}{2(n+2)} + \frac{1}{2(n+3)} \end{aligned}$$

Notes

(i) Avoid writing $S = \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) + \dots$, as it complicates the cancelling.]

(ii) The method of differences relies on there being a suitable form for the partial fractions!