STEP - Transformations (6 pages; 12/12/19)

(1) Translation of $\binom{a}{b}$: $y = f(x) \rightarrow y - b = f(x - a)$

(2) Stretch of scale factor k in the x direction (eg if $k = 2$, graph of $y = x^2$ is stretched outwards, so that the x-coordinates are doubled): $y = f(x) \rightarrow y = f(\frac{x}{k})$

Stretch of scale factor k in the y direction: $y = f(x) \rightarrow \frac{y}{k} = f(x)$

(3) Note that, at each stage of a composite transformation, we must be replacing x by either $x + a$ (where a can be negative) or kx (and similarly for y).

(4) Reflection in the line $x = L$: $y = f(x) \rightarrow y = f(2L - x)$

Reflection in the line $y = L$: $y = f(x) \rightarrow 2L - y = f(x)$

Special cases:

Reflection in the line $x = 0$: $f(x) \rightarrow f(-x)$

Reflection in the line $y = 0$: $y = f(x) \rightarrow -y = f(x)$

See also Appendix.

(5) **Example**: Applying stretches of scale factors a & b in the x & y directions to the circle $x^2 + y^2 = 1$ gives the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

(6) **Example**: To obtain $y = \sin(2x + 60)$ from $y = sinx$,

either (a) stretch by scale factor $\frac{1}{2}$ in the x direction, to give

$y = \sin(2x)$, and then translate by $\binom{-30}{0}$, to give

$y = \sin(2[x + 30]) = \sin(2x + 60)$

or (b) translate by $\binom{-60}{0}$, to give $y = \sin(x + 60)$, and then

stretch by scale factor $\frac{1}{2}$ in the x direction, to give

$y = \sin(2x + 60)$ [It is perhaps more awkward to produce a sketch by method (b).]

[Note that, at each stage, we are either replacing x by kx, or by

$x \pm a$]

(7) Transformations involving moduli signs

(i) $y = f(|x|)$

$f(|x|) = f(x)$ when $x \geq 0$

$f(|x|) = f(-x)$ when $x < 0$ (ie the left-hand half of $y = f(x)$ is replaced by the reflection of the right-hand half in the y-axis)

(ii) $|y| = f(x)$

As $|y| \geq 0$, the graph is undefined where $f(x) < 0$.

Where $f(x) \geq 0$, the graph of $|y| = f(x)$ is that of $y = f(x)$, together with its reflection in the x-axis.

(8) Rotation about general point

A rotation of $\theta°$ [anti-clockwise](of eg a shape) about a point (a,b) is equivalent to a translation $\binom{-a}{-b}$, followed by a rotation of $\theta°$ about the origin, and a translation of $\binom{a}{b}$. (*)

When $\theta = 180$, another equivalent combination of transformations is a reflection in the line $x = a$, followed by a reflection in the line $y = b$. $(**)$

Special case: A rotation of 180° is equivalent to a reflection in the line $x = 0$, followed by a reflection in the line $y = 0$, so that

$y = f(x) \to y = -f(-x)$

Exercise: Demonstrate the equivalence of (*) and (**) for the general function $y = f(x)$ (when $\theta = 180$).

Solution

For (*):

$y = f(x) \to y + b = f(x + a)$ [translation of $\binom{-a}{-b}$]

$\to -y + b = f(-x + a)$ [reflection in $x = 0$ & reflection in $y = 0$; equivalent to rotation of 180° about the origin]

$\to -(y - b) + b = f(-[x - a] + a)$; ie $2b - y = f(2a - x)$

For (**):

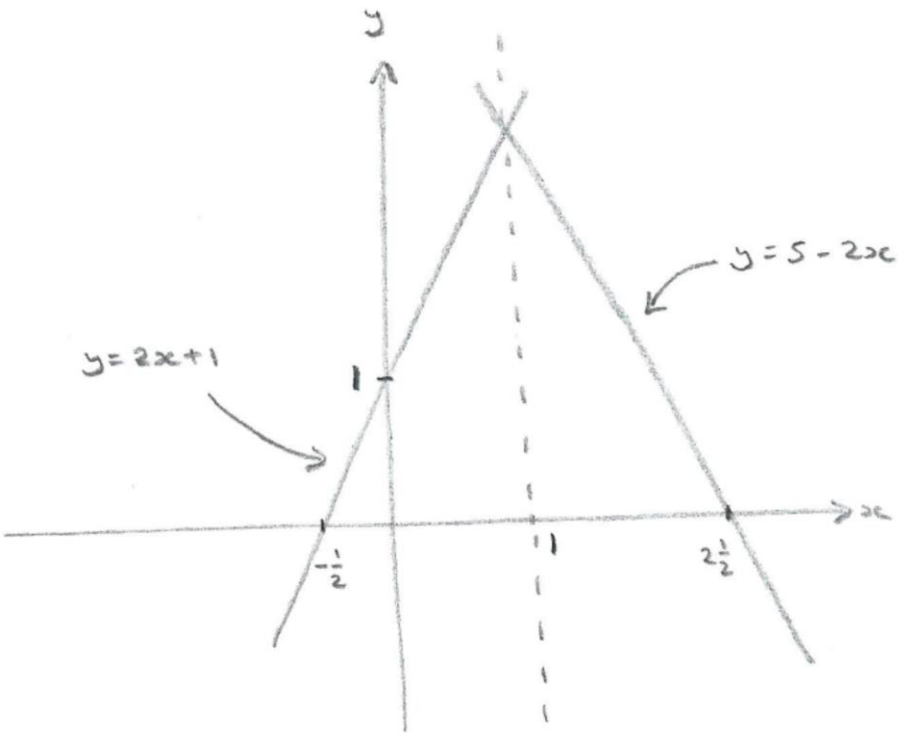
For a reflection in the line $x = a$, x is replaced by $2a - x$ (see separate note on reflection in line $x = L$). Similarly, for a reflection in $y = b$; giving $2b - y = f(2a - x)$, as above.

Appendix: Reflection in line $x = L$: Example & Exercises

Example: Reflect the line $y = 2x + 1$ in the line $x = 1$

Method A

First reflect $y = 2x + 1$ in the line $x = 0$. The image intersects the line $y = 2x + 1$ at $(0,1)$. For a reflection in the line $x = 1$, we want the image of $(0,1)$ to be at $(2,1)$. So a translation of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is required.



Algebraically, $y = 2x + 1$ is first transformed to

$y = 2(-x) + 1 = 1 - 2x$;

then to $y = 1 - 2(x - 2) = 5 - 2x$

Method B

The new line will have gradient -2, and meet the line $y = 2x + 1$ when $x = 1$; ie at the point $(1, 3)$.

So it has equation $\frac{y-3}{x-1} = -2$; ie $y - 3 = -2x + 2$, or $y = 5 - 2x$

(obviously method A can be applied more generally then B)

Method C

In order to reflect the curve $y = f(x)$ in the line $x = L$, to give the function $y = g(x)$, we require $g(L + u) = f(L - u)$

Let $x = L + u$, so that $L - u = L - (x - L) = 2L - x$

Thus, for a reflection in the line $x = L$, $f(x)$ is transformed to $f(2L - x)$.

This is equivalent to a reflection in $x = 0$, followed by a translation of $\binom{2L}{0}$: $f(x) \rightarrow f(-x) \rightarrow f(-[x - 2L]) = f(2L - x)$.

Note: The result $sin(\pi - x) = sinx$ follows from the fact that $y = sinx$ is symmetrical about $x = \frac{\pi}{2}$.

Exercise 1: Find the function resulting from reflecting $y = cosx$ in the line $x = \frac{\pi}{2}$, and then in the line $y = 1$

Solution

For a reflection in the line $x = \frac{\pi}{2}$, first reflect in $x = 0$:

$y = cosx \rightarrow y = \cos(-x) = cosx$;

then translate by $\begin{pmatrix}\pi\\0\end{pmatrix} \rightarrow y = \cos(x - \pi) = cosxcos\pi + sinxsin\pi$;

ie $y = -cosx$

Then for a reflection in the line $y = 1$, first reflect in $y = 0$:

$y = -cosx \rightarrow -y = -cosx; ie\ y = cosx$;

and then translate by $\begin{pmatrix}0\\2\end{pmatrix}$, to give $y - 2 = cosx$;

ie $y = cosx + 2$

Exercise 2

Symmetry in $x = L$ occurs when $f(L - x) = f(L + x)$

Show that this is equivalent to $f(2L - x) = f(x)$.

Solution

If $f(L - x) = f(L + x)$, then

$f(2L - x) = f(L + [L - x]) = f(L - [L - x]) = f(x)$

Conversely, if $f(2L - x) = f(x)$, then

$f(L - x) = f(2L - L - x) = f(2L - [L + x]) = f(L + x)$

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(4) Example: The graph of $y = \frac{x-2}{x-1}$ can be obtained from that of $y = \frac{1}{x}$ by a sequence of transformations:

First of all, $\frac{x-2}{x-1} = 1 - \frac{1}{x-1}$

Starting with $y = \frac{1}{x}$,

(i) translation of $\binom{1}{0}$, to give $y = \frac{1}{x-1}$

(ii) reflection in the x axis, to give $y = -\frac{1}{x-1}$

(iii) translation of $\binom{0}{1}$, to give $y = 1 - \frac{1}{x-1}$

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(6) Example: To transform $y = sinx$ to $y = \sin(2x - \frac{\pi}{3})$,

Either (a) translate by $\begin{pmatrix} \frac{\pi}{3} \\ 0 \end{pmatrix}$, to give $y = \sin\left(x - \frac{\pi}{3}\right)$, and then stretch by factor $\frac{1}{2}$, to give $y = \sin(2x - \frac{\pi}{3})$,

or (b) stretch by factor $\frac{1}{2}$, to give $y = \sin(2x)$, and then translate by $\begin{pmatrix} \frac{\pi}{6} \\ 0 \end{pmatrix}$, to give $y = \sin\left(2\left[x - \frac{\pi}{6}\right]\right) = \sin(2x - \frac{\pi}{3})$.

[Option (b) may be easier to carry out.]

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(v) Reflection in line $x = L$

Consider a point with x coordinate $L + \delta$. A reflection in the line $x = L$ can be achieved by reflecting in the y-axis, and then translating by $\binom{2L}{0}$. So

$y = f(x)$ becomes $y = f(-x)$, and then $y = f(-[x - 2L]) = f(2L - x)$

Example: A reflection in the line $x = \frac{\pi}{2}$ transforms $y = sinx$ to

$y = \sin(\pi - x)$ (which is the same function, as $y = sinx$ is symmetric about $x = \frac{\pi}{2}$).

Similarly for a reflection in $y = L$.