STEP - Series

Triangular numbers are defined as follows:

$T_r = \frac{1}{2}r(r+1)$ for integer $r \geq 1$

Prove that $\sum_{r=1}^{\infty} \frac{1}{T_r} = 2$

Solution

$\sum_{r=1}^{\infty} \frac{1}{T_r} = 2\sum_{r=1}^{\infty} \frac{1}{r(r+1)} = 2\sum_{r=1}^{\infty} (\frac{1}{r} - \frac{1}{r+1})$

$= 2\left\{\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots\right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots\right)\right\} = 2$

Perfect powers are defined as follows:

m^k for integer $m \geq 2$ & integer $k \geq 2$

Prove that the sum of the reciprocals of all perfect powers is 1 (including duplicates; eg $4^2 = 2^4$).

Solution

$\sum_{m=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{m^k} = \sum_{m=2}^{\infty}\frac{\frac{1}{m^2}}{1-\frac{1}{m}} = \sum_{m=2}^{\infty}\frac{1}{m(m-1)} = \sum_{m=2}^{\infty}(\frac{1}{m-1} - \frac{1}{m})$

$= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots\right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots\right) = 1$

$\sum_{r=1}^{\infty} r a^r$

Solution

$\sum_{r=1}^{\infty} ra^r = a\frac{d}{da}\sum_{r=1}^{\infty} a^r = a\frac{d}{da}(\frac{a}{1-a})$ (when $|a| < 1$)

$= a\,.\frac{(1-a)-a(-1)}{(1-a)^2} = \frac{a}{(1-a)^2}$