STEP - Reference (9 pages; 3/7/25)

Contents

Cubics

For the cubic $f(x) = ax^3 + bx^2 + cx + d$:

(i) There is always one point of inflexion, at $x = -\frac{b}{3a}$

(ii) Cubic curves have rotational symmetry (of order 2) about the PoI.

(iii) The (x-coordinate of the) PoI lies midway between any turning points.

(iv) The (x-coordinate of the) PoI is the average of the roots, when there are 3 real roots (and also when there are complex roots).

(v) There will be two turning points when $b^2 > 3ac$

Derivatives [See Differentiation I&E for proofs]

$\frac{d}{dx}(a^x) = lna.a^x$

$\frac{d}{dx}(x^x) = x^x(1 + lnx)$

$\frac{d}{dx}\left(x^{sinx}\right) = x^{sinx}(\frac{1}{x}sinx + lnx.cosx)$

$\frac{d}{dx}log_a x = \frac{1}{xlna}$

Expansions

(1) (i) $(a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + ac + bc)$

(ii) $(a + b + c)^3 = (a^3 + b^3 + c^3)$

$+3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)$

$+6abc$

(iii) $(a + b + c)^4 = (a^4 + b^4 + c^4)$

$+4(a^3b + a^3c + b^3a + b^3c + c^3a + c^3b)$

$+6(a^2b^2 + a^2c^2 + b^2c^2) + 12(a^2bc + b^2ac + c^2ab)$

(iv) $(a+b+c)^n = \sum_{\substack{i,j,k \\ (i+j+k=n)}} \binom{n}{i,j,k} a^i b^j c^k$,

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$

(2) Taylor expansions

(i) Maclaurin: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$

(ii) Taylor I: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$

(iii) Taylor II: $f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \cdots$

[$x = 0$ gives the Maclaurin expansion]

Factors

Let $f(n)$ be the number of factors of n (including 1).

If $n = pq$, where p & q have no common factors (other than 1), then $f(n) = f(p)f(q)$.

[eg $100 = 2^2 \times 5^2$; factors are obtained from $\{1, 2, 4\}$ with $\{1, 5, 25\}$, giving a total of $3 \times 3 = 9$ factors: $1, 5, 25, 2, 10, 50, 4, 20, 100$]

Factorisations (algebraic)

(i) $x^2 - y^2 = (x-y)(x+y)$

(ii) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

[Let $f(x) = x^3 - y^3$. Then $f(y) = 0$, and so $x - y$ is a factor of $x^3 - y^3$, by the Factor Theorem.]

(iii) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

or $(x+y)(x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1})$, if n is even

$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$ if n is odd

Functions

(1) Reflection in the line $x = a$: $f(x) \to f(2a - x)$

Geometry & Solids

(1) Tangents and normals to conics

(i) Parabola $y^2 = 4ax$ at $(at^2, 2at)$

tangent: $y = \frac{1}{t}x + at$

normal: $y = -tx + 2at + at^3$

(ii) Rectangular hyperbola $xy = c^2$

tangent: $y = -\frac{1}{t^2}x + \frac{2c}{t}$

normal: $y = t^2x + \frac{c}{t} - ct^3$

(2) Areas & Volumes

(i) Area of sector: $\frac{1}{2}r^2\theta$

(consider limit of area of triangle $\frac{1}{2}r^2 sin\theta$ as $\theta \to 0$)

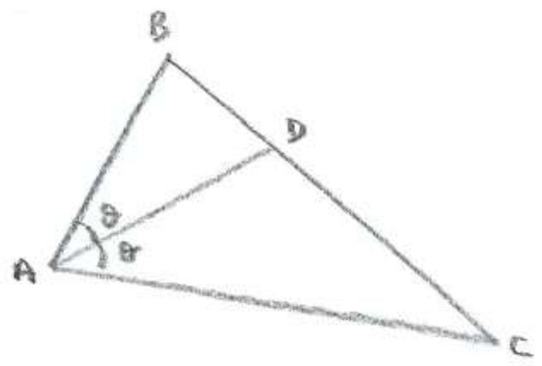
(ii) Volume of sphere: $\frac{4}{3}\pi r^3$

(iii) Volume of pyramid or cone: $\frac{1}{3} \times$ base area $\times$ height

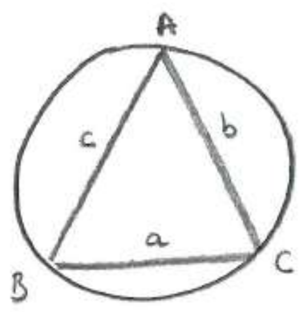
(3) Angle Bisector Theorem

Referring to the diagram below, the Angle Bisector theorem says that

$\frac{BD}{DC} = \frac{AB}{AC}$



(4) ABC is a triangle circumscribed by a circle of radius R, as shown in the diagram below.



As an extension of the Sine rule, $\frac{a}{sinA} = 2R$

(5) Heron's formula for the area of a triangle with sides a, b & c:

$\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$

Hyperbolic Functions

$cosh^2x + sinh^2x = cosh2x$; $cosh^2x - sinh^2x = 1$

$arsinhx = \ln\left(x + \sqrt{x^2 + 1}\right)$; $arcoshx = \ln\left(x + \sqrt{x^2 - 1}\right)$ $(x \geq 1)$

$artanhx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $(|x| < 1)$

Limits

Note that $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$ only when $\lim_{x\to\infty} f(x)$ & $\lim_{x\to\infty} g(x)$ are constants. [See "A course of Pure Mathematics" by G.H. Hardy (CUP 1933): Theorem IV of section 66]

Example where $\frac{\lim f(x)}{\lim g(x)} \neq lim\frac{f(x)}{g(x)}$:

$y = \frac{6x^2-5x+3}{3x-1} = \cdots = 2x - 1 + \frac{2}{3x-1} \to 2x - 1$

[taking $f(x) = \frac{2}{3x-1}$ and $g(x) = 1$, and noting that $\lim_{x\to\infty} f(x) = 0$]

but $\frac{6x-5+3/x}{3-1/x}$ wrongly suggests a limit of $\frac{6x-5}{3} = 2x - \frac{5}{3}$

Numbers

Golden ratio: $\frac{1+\sqrt{5}}{2} = 1.618$ (3dp)

Numerical Methods

(1) Simpson's rule

$\int_a^b y\,dx \approx$

$$\frac{h}{3}\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}$$

where $h = \frac{b-a}{n}$ (n even)

Polynomials

(1) Integer roots

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$

where $n \geq 2$ and the a_i are integers, with $a_0 \neq 0$.

Then it can be shown that any rational root of the equation $f(x) = 0$ will be an integer.

Proof

Suppose that there is a rational root $\frac{p}{q}$, where p & q are integers with no common factor greater than 1 and $q > 0$.

Then $\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_2\left(\frac{p}{q}\right)^2 + a_1\left(\frac{p}{q}\right) + a_0 = 0$

and, multiplying by q^{n-1}:

$\frac{p^n}{q} + a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1pq^{n-2} + a_1q^{n-1} = 0$

Then, as all the terms from $a_{n-1}p^{n-1}$ onwards are integers, it follows that $\frac{p^n}{q}$ is also an integer, and hence $q = 1$ (as p & q have no common factor greater than 1), and the root is an integer.

Series

(1) $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$; $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$

Trigonometry

(1) Powers of Sines and Cosines

[To derive $sin^n\theta$ from $cos^n\theta$, write $sin^n\theta = cos^n(90-\theta)$ etc]

$cos^3\theta = \frac{1}{4}(cos3\theta + 3cos\theta)$

$sin^3\theta = \frac{1}{4}(-sin3\theta + 3sin\theta)$

$cos^4\theta = \frac{1}{8}(cos4\theta + 4cos2\theta + 3)$

$sin^4\theta = \frac{1}{8}(cos4\theta - 4cos2\theta + 3)$

$cos^5\theta = \frac{1}{16}(cos5\theta + 5cos3\theta + 10cos\theta)$

$sin^5\theta = \frac{1}{16}(sin5\theta - 5sin3\theta + 10sin\theta)$

$cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$

$sin^6\theta = \frac{1}{32}(-cos6\theta + 6cos4\theta - 15cos2\theta + 10)$

$cos^7\theta = \frac{1}{64}(cos7\theta + 7cos5\theta + 21cos3\theta + 35cos\theta)$

$sin^7\theta = \frac{1}{64}(-sin7\theta + 7sin5\theta - 21sin3\theta + 35sin\theta)$

(2) $cos(n\theta), sin(n\theta)$

[sin($2m\theta$) can't be expressed in terms of powers of $sin\theta$]

$cos3\theta = 4cos^3\theta - 3cos\theta$

$sin3\theta = -4sin^3\theta + 3sin\theta$

$cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$

$cos5\theta = 16cos^5\theta - 20cos^3\theta + 5cos\theta$

$sin5\theta = 16sin^5\theta - 20sin^3\theta + 5sin\theta$

$cos6\theta = 32cos^6\theta - 48cos^4\theta + 18cos^2\theta - 1$

$$cos7\theta = 64cos^7\theta - 112cos^5\theta + 56cos^3\theta - 7\cos\theta$$

$$sin7\theta = -64sin^7\theta + 112sin^5\theta - 56sin^3\theta + 7sin\theta$$