STEP: Rearrangements – Ideas & Exercises

(13 pages; 4/7/25)

(1) Rearranging an expression or equation

eg in order for an earlier result to be used

$\int \frac{1+x}{x-1}\, dx$

$\int \frac{1+x}{x-1}\,dx = \int \frac{x-1}{x-1}\,dx + \int \frac{2}{x-1}\,dx$

(2) Graphical v algebraic approach

eg integer $x, y \rightarrow$ points on a grid

How many solutions are there to $x^3 - 6x^2 + 9x + 2 = 0$?

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Solution

$\Leftrightarrow x(x^2 - 6x + 9) = -2$

$\Leftrightarrow x(x-3)^2 = -2$

So one solution, from graph of $y = x(x-3)^2$

(3) Substitution

Solve the equation $x - \sqrt{x} = 6$

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Solution

Method 1

Let $y = \sqrt{x}$, so that

$x - \sqrt{x} - 6 = 0 \Rightarrow y^2 - y - 6 = 0$

$\Rightarrow (y + 2)(y - 3) = 0$

$\Rightarrow y = -2$ (reject as $\sqrt{x}$ must be ≥ 0) $or\ y = 3$

Method 2

Let $f(x) = x - \sqrt{x} - 6$

$f(x) = 0 \Rightarrow x - 6 = \sqrt{x}$

$\Rightarrow (x - 6)^2 = x$, but this may include spurious solutions

[of $x - 6 = -\sqrt{x}$]

$\Rightarrow x^2 - 13x + 36 = 0$

$\Rightarrow (x - 9)(x - 4) = 0$

$\Rightarrow x = 9\ or\ x = 4$

$f(9) = 0$ & $f(4) = -4$

Thus the only solution is $x = 9$

[Let $g(x) = x + \sqrt{x} - 6 = 0$

Then $g(x) = 0 \Rightarrow (x - 6)^2 = x$ as well

$g(9) \neq 0, and\ g(4) = 0$]

Find $\frac{d}{dx}(a^x)$

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Solution

Method 1

Let $a = e^b$. Then $\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{bx}) = be^{bx} = lna.a^x$

Method 2

Let $y = a^x$. Then $lny = xlna$,

Find all positive integer solutions of the equation

$xy - 8x + 6y = 90$

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Solution

[Aiming for something of the form $f(x)g(y) = c$, where c is an integer:]

$xy - 8x + 6y = (x + 6)(y - 8) + 48,$

so that the original equation is equivalent to

$(x + 6)(y - 8) = 42$

The positive integer solutions are given by:

$x + 6 = 7, y - 8 = 6$

$x + 6 = 14, y - 8 = 3$

$x + 6 = 21\ y - 8 = 2$

$x + 6 = 42, y - 8 = 1,$

so that the solutions are:

$x = 1, y = 14$

$x = 8, y = 11$

$x = 15, y = 10$

$x = 36, y = 9$

Can n^3 equal $n + 12345670$ (where n is a positive integer)?

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Solution

Rearrange to $n^3 - n = 12345670$

$n^3 - n = n(n^2 - 1) = n(n-1)(n+1)$, and one of these factors must be a multiple of 3; whereas 12345670 is not a multiple of 3 (since $1 + 2 + 3 + 4 + 5 + 6 + 7 + 0$ isn't a multiple of 3); so answer is No.

From 2023/P3/Q7:

(iv) Let k be a continuous function defined for $0 \leqslant x \leqslant 1$ and a be a real number, such that

$$\int_0^1 \mathrm{e}^{ax}\left(\mathrm{k}(x)\right)^2 \,\mathrm{d}x = 2\int_0^1 \mathrm{k}(x)\,\mathrm{d}x + \frac{\mathrm{e}^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4}.$$

Show that a must be equal to 2 and find k.

Solution

Consider $\int_0^1 (e^{\frac{ax}{2}}k(x) - e^{-\frac{ax}{2}})^2 dx$

$= \int_0^1 e^{ax}(k(x))^2 dx + \int_0^1 e^{-ax}\, dx - 2\int_0^1 k(x) dx$

$= \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} + [-\frac{1}{a}e^{-ax}]_0^1$

$= \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} - \frac{1}{a}(e^{-a} - 1) = -\frac{1}{a^2} - \frac{1}{4} + \frac{1}{a}$

$= -\frac{(4+a^2-4a)}{4a^2}$

$= -\frac{(a-2)^2}{4a^2}$

Now $\int_0^1 (e^{\frac{ax}{2}}k(x) - e^{-\frac{ax}{2}})^2 dx \geq 0$, whilst $= -\frac{(a-2)^2}{4a^2} \leq 0$,

and so $\frac{(a-2)^2}{4a^2} = 0$, and therefore $a = 2$,

and $\int_0^1 (e^{\frac{ax}{2}}k(x) - e^{-\frac{ax}{2}})^2 dx = 0$,

so that $e^{\frac{ax}{2}}k(x) - e^{-\frac{ax}{2}} = 0$,

and hence $k(x) = e^{-ax} = e^{-2x}$.