Probability Methods

Approaches

(1) 'One step at a time'

eg P(drawing a red ball, followed by a blue ball)

$= P$(drawing a red ball on the 1st go)

$\times P$(drawing a blue ball on the 2nd go |a red ball was drawn 1st)

(2) Break down into cases:

$P(E) = P(A_1)P(E|A_1) + P(A_2)P(E|A_2)$

(3) Conditional probability: $P(E|A) = \frac{P(A\&E)}{P(A)}$

[often $P(A\&E) = P(E)$; eg if A is the event of rolling an even number on a die, and E is the event of rolling a 2]

(4) 'Basic Definition': $P(E) = \frac{number\ of\ outcomes\ where\ E\ occurs}{number\ of\ possible\ outcomes}$

provided that the outcomes are equally likely

Notes

(i) The outcomes will be equally likely when $\binom{n}{r}$ is employed.

(ii) Given the choice, it will normally be easier to count outcomes on the basis that order isn't important.

(iii) Special case: $P(E|A) = \frac{number\ of\ outcomes\ where\ A\ \&\ E\ occur}{number\ of\ outcomes\ where\ A\ occurs}$

(5) Recurrence relation

A and B take it in turns to shoot arrows at a target, with A starting first. The probability that A hits the target is a and the probability that B hits the target is b. The winner is the person who hits the target first. Find the probability that A wins.

Solution

Let α be the probability that A wins.

Then $\alpha = \text{P(A wins on 1st attempt)} + \text{P(wins after 1st attempt)}$

[Case by Case approach]

$= a + (1-a)(1-b)\alpha$, **[Recurrence relation]**

so that $\alpha(1-(1-a)(1-b)) = a$,

and $\alpha = \frac{a}{1-(1-a)(1-b)} = \frac{a}{a+b-ab}$

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Solution

Approach 1: 'One step at a time'

P(3 different letters are chosen)

$= \text{P(any letter is chosen initially)}$

$\times \text{P(a different letter is then chosen)}$

$\times \text{P(a letter different from the 1st 2 is then chosen)}$

$= 1 \times \frac{6}{8} \times \frac{3}{7} = \frac{9}{28}$

Approach 2: Consider 1 way & count number of ways

$\text{P(ABC are chosen} - \text{in that order)} = \frac{3}{9} \times \frac{3}{8} \times \frac{3}{7} = \frac{3}{7\times 8}$

As there are 3! ways of ordering ABC,

$\text{P(3 different letters are chosen)} = \frac{3}{7\times 8} \times 3! = \frac{9}{28}$

Approach 3a: 'Basic definition'

P(3 different letters are chosen)

$$= \frac{\text{no. of ways of obtaining the letters ABC (where order doesn't matter)}}{\text{no. of ways of chosing 3 letters out of 9 (where order doesn't matter)}}$$

$$= \frac{\binom{3}{1}\times\binom{3}{1}\times\binom{3}{1}}{\binom{9}{3}} = \frac{27}{\left(\frac{9(8)(7)}{3!}\right)} = \frac{9}{28}$$

Approach 3b:

P(3 different letters are chosen)

$$= \frac{\text{no. of ordered selections involving the letters ABC}}{\text{total no. of ordered selections}}$$

$$= \frac{\text{no. of ways of obtaining ABC (in that order)} \times 3!}{9 \times 8 \times 7} = \frac{3 \times 3 \times 3 \times 3!}{9 \times 8 \times 7} = \frac{9}{28}$$

Devices

(i) Symmetry

When throwing a fair die repeatedly:

P(At least one 5 arises before the 1st 6)

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$= P$(a 5 occurs before a 6) $= \frac{1}{2}$

(ii) $P(at\ least\ 1) = 1 - P(none)$

Similarly: initially include non-permissible cases, and then deduct them

(iii) eg $P(3\ As\ \&\ 2\ Bs)$

$= P(AAABB) \times number\ of\ arrangements\ of\ 3\ As\ \&\ 2\ Bs$