STEP : Necessary & Sufficient Conditions: Ideas & Exercises (14 pages; 13/8/25)

The following are equivalent:

$(X) \Rightarrow (Y)$ [(X) implies (Y)]

(X) is a sufficient condition for (Y)

(Y) is a necessary condition for (X)

(X) is true only if (Y) is true [(X) cannot be true if (Y) isn't true]

(Y) is true if (X) is true

Example

(X): The equation $ax^2 + bx + c = 0$ has a real solution

(Y): $b^2 - 4ac \geq 0$

Then (X) $\Leftrightarrow (Y)$

The following are equivalent:

(X) $\Leftrightarrow (Y)$ [(X) implies and is implied by (Y)]

(X) is true if and only if (Y) is true

[ought to be "(X) is true only if and if (Y) is true"]

(X) is a necessary & sufficient condition for (Y)

[ought to be "(X) is a sufficient & necessary condition for (Y)"]

Find a simple constraint on a and/or b such that it is true that

$a > b \Leftrightarrow a^2 > b^2$

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Solution

We can consider the graph of $y = x^2$.

$a > b \Rightarrow a^2 > b^2$ when $b \geq 0$ or when $b < 0$ and $a > |b|$

$a^2 > b^2 \Rightarrow a > b$ only when $a > 0$

A simple constraint would be: $b \geq 0$ (then $a > b \Rightarrow a > 0$).

Example

Living in London ⇒ Living in England (*)

but Living in England ⇏ Living in London

[ie the 'converse' of (*): "Living in England ⇒ Living in London" is not true]

"Living in London" is a sufficient condition for "living in England"

"Living in England" is a necessary condition for "living in London"

"Living in England" is not a sufficient condition for "living in London"

"Living in London" is not a necessary condition for "living in England"

'Contrapositive' of $(X) \Rightarrow (Y)$ is $(Y') \Rightarrow (X')$

The two implications are mathematically equivalent.

So (in the above example), $(L) \Rightarrow (E)$ is equivalent to $(E') \Rightarrow (L')$.

(The set (L) is contained within the set (E).)

Possible approaches to proving that ($\boldsymbol{X}$) $\Leftrightarrow$ ($\boldsymbol{Y}$)

(1) $(X) \Leftrightarrow (A) \Leftrightarrow (B) \Leftrightarrow (Y)$

(2) $(X) \Leftrightarrow (A) \Leftrightarrow (Z)$ and $(Y) \Leftrightarrow (B) \Leftrightarrow (Z)$

(3) $(X) \Rightarrow (Y)$ and $(Y) \Rightarrow (X)$

(4) $(X) \Rightarrow (Y)$ and $(X') \Rightarrow (Y')$

(whenever X is true, Y is true; and whenever X isn't true, Y isn't true) [$X' \Rightarrow Y'$ is the 'inverse' of $X \Rightarrow Y$]

(5) Break down into cases, and show that $(X) \Rightarrow (Y)$ and $(X') \Rightarrow (Y')$ in each case (ie either (X) and (Y) are both true, or (X) and (Y) are both false).

Note: If we only need to prove that $(Y) \Rightarrow (X)$, it may be easiest to prove that $(X) \Leftrightarrow (Y)$, and then deduce that $(Y) \Rightarrow (X)$.

What is wrong with the following?

$x = 1 \Rightarrow x - 1 = 0 \Rightarrow (x - 1)(x - 2) = 0 \Rightarrow x = 1 \; or \; x = 2$

So x can be either 1 or 2.

$x = 1 \; or \; x = 2$ means that $x \; \varepsilon \; (1,2)$;

ie x must be either 1 or 2 (not "x CAN be either 1 or 2")

[$x = 2$ is a 'spurious' solution]

$x = 1 \; or \; x = 2$ [ie for any x such that $x \; \varepsilon \; (1,2)$] $\Rightarrow$

$(x-1)(x-2) = 0 \nRightarrow x - 1 = 0$;

so $x = 1 \Rightarrow x = 1 \; or \; x = 2$

but $x = 1 \; or \; x = 2 \nRightarrow x = 1$

Rewrite these two statements using the wording "necessary condition" and/or "sufficient condition".

$\boldsymbol{x = 1 \Rightarrow x = 1 \; or \; x = 2}$

$x = 1$ is a sufficient condition for $x = 1 \; or \; x = 2$

$x = 1 \; or \; x = 2$ is a necessary condition for $x = 1$

$\boldsymbol{x = 1 \; or \; x = 2 \nRightarrow x = 1}$

$x = 1 \; or \; x = 2$ is not a sufficient condition for $x = 1$

$x = 1$ is not a necessary condition for $x = 1 \; or \; x = 2$

Given that a, b & c are positive numbers, prove that $\frac{a}{b} < \frac{a+c}{b+c}$ when $a < b$.

[Given that a, b & c are positive numbers, prove that $\frac{a}{b} < \frac{a+c}{b+c}$ when $a < b$.]

What is wrong with the following:

$\frac{a}{b} < \frac{a+c}{b+c} \Rightarrow a(b+c) < b(a+c)$ (as $b > 0$ & $b + c > 0$)

$\Rightarrow ac < bc \Rightarrow a < b$ (as c>0)

[Given that a, b & c are positive numbers, prove that $\frac{a}{b} < \frac{a+c}{b+c}$ when $a < b$.]

The above argument is a proof that

$\frac{a}{b} < \frac{a+c}{b+c} \Rightarrow a < b$; not that $a < b \Rightarrow \frac{a}{b} < \frac{a+c}{b+c}$

What is wrong with the following:

$\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a(b+c) < b(a+c)$ (as $b > 0$ & $b + c > 0$)

$\Leftrightarrow ac < bc \Leftrightarrow a < b$ (as c>0)

[Given that a, b & c are positive numbers, prove that $\frac{a}{b} < \frac{a+c}{b+c}$ when $a < b$.]

We need to make it clear that $a < b \Rightarrow \frac{a}{b} < \frac{a+c}{b+c}$ is the required result.

Improved solution:

$\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a(b+c) < b(a+c)$ (as $b > 0$ & $b + c > 0$)

$\Leftrightarrow ac < bc \Leftrightarrow a < b$ (as c>0)

So $a < b \Rightarrow \frac{a}{b} < \frac{a+c}{b+c}$; ie $\frac{a}{b} < \frac{a+c}{b+c}$ when $a < b$, as required.

If n is a positive integer, and n^2 is odd (A), prove that n is odd (B).
[Result to prove: $A \Rightarrow B$]

If n is a positive integer, and n^2 is odd (A), prove that n is odd (B).
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Solution

Method 1: Proof by contradiction

Suppose that n is even. Then $n = 2m$, for some positive integer m.

But then $n^2 = (2m)^2 = 4m^2$, which is divisible by 2, and hence even. This contradicts the fact that n^2 is odd, and so n must be odd.

Method 2: Using contrapositive

To prove that $B' \Rightarrow A'$:

Suppose that n is even. Then (as before) n^2 is even, so that A' holds.