STEP - Matrices

Consider the transformation represented by $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

(i) What is the significance of $\begin{pmatrix} a \\ b \end{pmatrix}$?

(ii) What happens when (a) $|M| = 1$? (b) $|M| = 0$?

A transformation is represented by $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where $c \neq 0$

(i) Show that the condition for the existence of a line of invariant points is that $tr(M) = |M| + 1$

[A line of invariant points is one for which all points on the line transform to themselves. The trace of M, $tr(M) = a + d$.]

Solution

Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Then $ap + cq = p$ & $bp + dq = q$,

so that $(a-1)p + cq = 0$ & $bp + (d-1)q = 0$

In order for there to be a solution other than $p = q = 0$,

$\begin{vmatrix} a-1 & c \\ b & d-1 \end{vmatrix} = 0$,

so that $(a-1)(d-1) - bc = 0$,

and $ad - bc - (a+d) + 1 = 0$;

giving $tr(M) = \mathrm{a} + \mathrm{d} = |M| + 1$

This argument is reversible, and so there will be a line of invariant points when $tr(M) = |M| + 1$

(ii) Show that the condition for the existence of invariant lines is that $[tr(M)]^2 \geq 4|M|$

[An invariant line is one for which all points on the line transform to a point on that line (ie transform to either themselves or another point on the line).]

[(ii) Show that the condition for the existence of invariant lines is that $[tr(M)]^2 \geq 4|M|$]

Solution

(i) Suppose that there is an invariant line $y = mx + k$.

Then $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} ax + cmx + ck \\ bx + dmx + dk \end{pmatrix}$

and $bx + dmx + dk = m(ax + cmx + ck) + k$

This must apply for all x, so:

equating coefficients of x: $b + dm = ma + cm^2$,

so that $cm^2 + m(a - d) - b = 0$ (1)

And equating constant terms gives $dk = mck + k$,

so that $k(d - mc - 1) = 0$ (2)

Then, in order for there to be an m that satisfies (1),

the discriminant of (1) must be non-negative (and vice-versa)

[noting that $c \neq 0$, so that (1) is a quadratic]

ie $(a - d)^2 + 4cb \geq 0$

$\Leftrightarrow (a + d)^2 - 4ad + 4cb \geq 0$

$\Leftrightarrow [tr(M)]^2 \geq 4|M|$, as required.

Note: It can also be shown that there will be a family of invariant lines if and only if $\boldsymbol{tr(M) = |M| + 1}$

(iii) What is special about $c = 0$?

[(iii) What is special about $c = 0$?]

For $M = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$, $\begin{pmatrix} 0 \\ d \end{pmatrix}$ is the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The y-axis will be an invariant line, but its gradient is undefined.

Example: $\begin{pmatrix} 4 & 1 \\ -9 & -2 \end{pmatrix}$

$\Delta = 1$

As $tr = \Delta + 1$, there is a line of invariant points (through the Origin), and a family of invariant lines.

It will in fact be a shear (with the line of invariant points being part of the family of invariant lines).

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$\Delta = 0 \Rightarrow$ images of all points lie on the same line:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \frac{p}{2} + \frac{q}{2} = u \ \& \ \frac{p}{2} + \frac{q}{2} = v \ ; v = u$$

so image line is $y = x$

To find the points that map to (u, u):

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ u \end{pmatrix} \Rightarrow \frac{x}{2} + \frac{y}{2} = u; y = 2u - x$$

Note that $y = 2u - x$ passes through (u, u).

So $y = 2u - x$ is a family of invariant lines.

Also, as $tr = \Delta + 1$, there is a line of invariant points (through the Origin).

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow \frac{p}{2} + \frac{q}{2} = p; q = p \ ;$$

ie line of invariant points is $y = x$

[Note: The line of invariant points isn't part of the family of invariant lines.]

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Idea for (iii) $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$: consider image(s) of particular point(s)

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$\Delta = -1$

As $tr = \Delta + 1$, there is a line of invariant points (through the Origin), and a family of invariant lines.

This suggests a reflection in $y = mx$.

If $y = mx$ is a line of invariant points, then

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix},$$

so that $-\frac{1}{2} + \frac{\sqrt{3}}{2}m = 1$ & $\frac{\sqrt{3}}{2} + \frac{m}{2} = m$; ie $m = \sqrt{3}$

Also, consider the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is $\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$.

We would expect $\left|\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}\right| = \left|\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|$, which it does.

Also, the angle 2θ between $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ is given by

$$(1)\left(-\frac{1}{2}\right) + (0)\left(\frac{\sqrt{3}}{2}\right) = \left|\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right| \left|\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}\right| cos2\theta,$$

so that $cos2\theta = -\frac{1}{2}$, and hence $\theta = \frac{\pi}{3}$,

so that a reflection in the line $y = mx$ would imply that

$m = tan\theta = \sqrt{3}$

Summary

Condition for existence of invariant lines: $[tr(M)]^2 \geq 4|M|$

Condition for existence of line of invariant points (and family of invariant lines): $tr(M) = |M| + 1$

When invariant lines exist:

If a line of invariant points doesn't exist:	One or two invariant lines through the Origin.
If a line of invariant points exists:	Family of invariant lines (including one through the Origin), and possibly a further invariant line through the Origin; one of the lines through the Origin will be a line of invariant points

Examples where line of invariant points exists:

$\begin{pmatrix} 4 & 9 \\ -1 & -2 \end{pmatrix}: \lvert M\rvert = 1; tr(M) = 2$	Shear (family of invariant lines, including a line of invariant points) [There will be a shear whenever $\lvert M\rvert = 1$ & $tr(M) = 2$]
$\begin{pmatrix} 4 & 6 \\ -2 & -3 \end{pmatrix}: \lvert M\rvert = 0; tr(M) = 1$	All points transform to a single line, which is a line of invariant points; further family of invariant lines [Note: the line of invariant points isn't part of the family here.]
$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}: \lvert M\rvert = -1; tr(M) = 0$	Reflection in $y = x$ (line of invariant points); family of invariant lines perpendicular to $y = x$

(i) Write the simultaneous equations

$ax + cy = e$ & $bx + dy = f$ in matrix form.

(ii) Condition for there to be a unique solution?

(iii) Condition for equations to be consistent?

[(i) Write the simultaneous equations

$ax + cy = e$ & $bx + dy = f$ in matrix form.]

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

(ii) Condition for there to be a unique solution?

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} \neq 0$$

(iii) Condition for equations to be consistent?

$\begin{vmatrix} a & c \\ b & d \end{vmatrix} \neq 0$ or $\begin{vmatrix} a & e \\ b & f \end{vmatrix} = 0$ [or alternatively $\begin{vmatrix} e & c \\ f & d \end{vmatrix} = 0$]

Is there a solution to the following equations?

$8x - 4z = 40$

$3x - 5y + z = 0$

$x - y = 2$

[Is there a solution to the following equations?

$8x - 4z = 40$

$3x - 5y + z = 0$

$x - y = 2$]

$\begin{vmatrix} 8 & 0 & -4 \\ 3 & -5 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 8(1) + (-4)(2) = 0$, so no unique solution

$\begin{vmatrix} 40 & 0 & -4 \\ 0 & -5 & 1 \\ 2 & -1 & 0 \end{vmatrix} = 40(1) + (-4)(10) = 0;$

so infinite number of solutions

[Using the 3rd eq'n to eliminate x,

$8(2 + y) - 4z = 40$ & $3(2 + y) - 5y + z = 0$;

ie $8y - 4z = 24$ & $-2y + z = -6$]