STEP: Introductory Ideas & Exercises (20 pages; 4/7/25)

Simplify the following:

(i) $27^{-\frac{2}{3}}$ (ii) $\cos(-210°)$ (iii) $log_4\left(\frac{1}{64}\right)$

(i) $27^{-\frac{2}{3}}$ (ii) $\cos(-210°)$ (iii) $log_4\left(\frac{1}{64}\right)$

Solution

(i) $\frac{1}{27^{\frac{2}{3}}} = \frac{1}{\left(27^{\frac{1}{3}}\right)^2} = \frac{1}{3^2} = \frac{1}{9}$

(ii) $\cos(-210°) = \cos(210°) = \cos(360° - 210°) = \cos(150°) = -\frac{\sqrt{3}}{2}$

(iii) $log_4\left(\frac{1}{64}\right) = log_4(4^{-3}) = -3$

(i) Does $\sqrt{4}$ equal 2 or ± 2? (ii) Simplify $\sqrt{x^2}$

(i) Does $\sqrt{4}$ equal 2 or ±2? (ii) Simplify $\sqrt{x^2}$

Solution

(1)(i) $\sqrt{4} = 2$ (consider quadratic formula) (ii) $\sqrt{x^2} = |x|$

Comment on the following: $\frac{1}{x} > 4 \Rightarrow 1 > 4x$

Solution

$\frac{1}{x} > 4 \Rightarrow 1 > 4x$

We don't know whether x is positive or negative (undefined for $x = 0$).

Evaluate $\int \frac{1+x}{x-1}\, dx$

Solution

$\int \frac{1+x}{x-1}\, dx = \int \frac{x-1}{x-1}\, dx + \int \frac{2}{x-1}\, dx$ etc

Prove that $sin^2\theta + cos^2\theta = 1$

Prove that $sin^2\theta + cos^2\theta = 1$

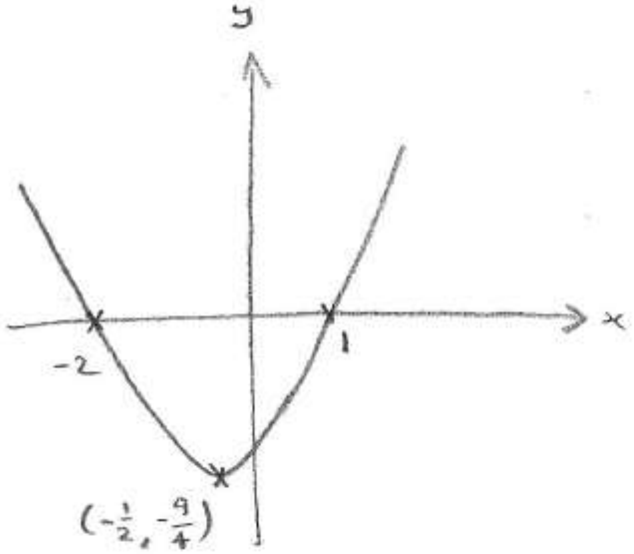
Solution

Apply Pythagoras to a right-angled triangle with sides cosθ, sinθ & 1.

Find the turning point of the graph of $y = (x - 1)(x + 2)$

Find the turning point of the graph of $y = (x-1)(x+2)$

Solution



Due to the symmetry of the curve about the vertical line through the turning point, the x-coordinate of the turning point will be $\frac{1}{2}(-2+1) = -\frac{1}{2}$

Then the y-coordinate is $= \left(-\frac{1}{2} - 1\right)\left(-\frac{1}{2} + 2\right) = \left(\frac{-3}{2}\right)\left(\frac{3}{2}\right) = -\frac{9}{4}$

Alternatively, we can complete the square:

$(x-1)(x+2) = x^2 + x - 2 = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - 2 = \left(x + \frac{1}{2}\right)^2 - \frac{9}{4}$

giving the turning point of $\left(-\frac{1}{2}, -\frac{9}{4}\right)$

Solve the equation $x - \sqrt{x} = 6$

Solve the equation $x - \sqrt{x} = 6$

Solution

Let $f(x) = x - \sqrt{x} - 6$

$f(x) = 0 \Rightarrow x - 6 = \sqrt{x}$

$\Rightarrow (x-6)^2 = x$, but this may include spurious solutions

[of $x - 6 = -\sqrt{x}$]

$\Rightarrow x^2 - 13x + 36 = 0$

$\Rightarrow (x-9)(x-4) = 0$

$\Rightarrow x = 9 \; or \; x = 4$

$f(9) = 0$ & $f(4) = -4$

Thus the only solution is $x = 9$

[Let $g(x) = x + \sqrt{x} - 6 = 0$

Then $g(x) = 0 \Rightarrow (x-6)^2 = x$ as well

$g(9) \neq 0, and \; g(4) = 0$]

Alternatively: Let $y = \sqrt{x}$, so that

$x - \sqrt{x} - 6 = 0 \Rightarrow y^2 - y - 6 = 0$

$\Rightarrow (y+2)(y-3) = 0$

$\Rightarrow y = -2$ (reject as $\sqrt{\mathrm{x}}$ must be ≥ 0) $or \; y = 3$

Composite transformations required to obtain

$y = \sin(2x + 60)$ from $y = sinx$?

Composite transformations required to obtain

$y = \sin(2x + 60)$ from $y = sinx$?

Solution

Either (a) stretch by scale factor $\frac{1}{2}$ i the x direction, to give

$y = \sin(2x)$, and then translate by $\begin{pmatrix} -30 \\ 0 \end{pmatrix}$, to give

$y = \sin(2[x + 30]) = \sin(2x + 60)$

or (b) translate by $\begin{pmatrix} -60 \\ 0 \end{pmatrix}$, to give $y = \sin(x + 60)$, and then

stretch by scale factor $\frac{1}{2}$ in the x direction, to give

$y = \sin(2x + 60)$

(It is perhaps more awkward to produce a sketch by method (b).)

How many solutions are there to $x^3 - 6x^2 + 9x + 2 = 0$?

How many solutions are there to $x^3 - 6x^2 + 9x + 2 = 0$?

Solution

$x^3 - 6x^2 + 9x + 2 = 0 \Leftrightarrow x(x^2 - 6x + 9) = -2$

$\Leftrightarrow x(x-3)^2 = -2$

So one solution, from graph of $y = x(x-3)^2$

Find the smallest positive integer n such that

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1}n \geq 100$$

Find the smallest positive integer n such that

$1-2+3-4+5-6+\cdots+(-1)^{n+1}n \geq 100$

Solution

[Presence of $(-1)^{n+1}$suggests considering even/odd n.]

With even n, the LHS becomes $1-2+3-4+\cdots-2m$, writing $n=2m$.

By grouping the terms as $(1-2)+(3-4)+\cdots-2m$, we see that this has a negative value.

So n must be odd, and the LHS becomes

$1-2+3-4+\cdots+(2m+1)$, writing $n=2m+1$

And the terms can be grouped to give

$(1-2)+(3-4)+\cdots([2m-1]-2m)+(2m+1)$

$=m(-1)+(2m+1)=m+1$

So we want $m+1 \geq 100$, and hence

$n=2m+1 \geq 2(99)+1=199$