STEP/Induction Q1 (18/6/23)

(i) If $y = e^x sinx$, show that $\frac{dy}{dx} = \sqrt{2}\, e^x \sin\,(x + \frac{\pi}{4})$

(ii) Prove by induction that $\frac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin\,(x + \frac{n\pi}{4})$

Solution

(i) $\frac{dy}{dx} = e^x sinx + e^x cosx = \sqrt{2}\, e^x\{sinx\left(\frac{1}{\sqrt{2}}\right) + cosx\left(\frac{1}{\sqrt{2}}\right)\}$

$= \sqrt{2}\, e^x\{sinxcos\left(\frac{\pi}{4}\right) + cosxsin\left(\frac{\pi}{4}\right)\}$

$= \sqrt{2}\, e^x \sin\,(x + \frac{\pi}{4})$

(ii) [Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$,

so that $\frac{d^k y}{dx^k} = \left(\sqrt{2}\right)^k e^x \sin\,(x + \frac{k\pi}{4})$

Then $\frac{d^{k+1}y}{dx^{k+1}} = \left(\sqrt{2}\right)^k e^x \sin\left(x + \frac{k\pi}{4}\right) + \left(\sqrt{2}\right)^k e^x \cos\,(x + \frac{k\pi}{4})$

$= \left(\sqrt{2}\right)^{k+1} e^x \{\sin\left(x + \frac{k\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \cos\,(x + \frac{k\pi}{4})\left(\frac{1}{\sqrt{2}}\right)\}$

$= \left(\sqrt{2}\right)^{k+1} e^x \{\sin\left(x + \frac{k\pi}{4}\right) cos\left(\frac{\pi}{4}\right) + \cos\,(x + \frac{k\pi}{4}) sin\left(\frac{\pi}{4}\right)\}$

$= \left(\sqrt{2}\right)^{k+1} e^x \sin\,(\left[x + \frac{k\pi}{4}\right] + \frac{\pi}{4})$

$= \left(\sqrt{2}\right)^{k+1} e^x \sin\,(x + \frac{(k+1)\pi}{4})$

[Standard wording]