Expectation & Variance

A pdf is defined as follows:

$P(X = x) = k(10 - x) \quad x = 0,1,2, \ldots,9$

$\qquad\qquad = 0 \qquad\qquad \text{otherwise}$

Find $E(X)$

$\sum_{x=0}^{9} k(10-x) = 1$

$\Rightarrow 10k(10) - k\sum_{x=1}^{9} x = 1$

$\Rightarrow k\left\{100 - \frac{1}{2}(9)(10)\right\} = 1$

$\Rightarrow k = \frac{1}{55}$

$E(X) = \sum_{x=0}^{9}\left(\frac{1}{55}\right)(10-x)x = \frac{10}{55}\sum_{0}^{9} x \; - \left(\frac{1}{55}\right)\sum_{x=0}^{9} x^2$

$= \frac{10}{55}\left(\frac{1}{2}\right)(9)(10) - \left(\frac{1}{55}\right)\left(\frac{1}{6}\right)(9)(10)(19)$

$= \frac{1}{55}(450 - 285) = \frac{165}{55} = \frac{33}{11} = 3$

$E[g(X)] = \sum_x g(x)P(X = x)$

$E[aX + b] = \sum_x (ax + b)P(X = x)$

$= [a \sum_x xP(X = x)] + b \sum_x P(X = x)$

$= aE(X) + b$

$Var(X) = E[(X - \mu)^2]$

Show that this can be written as $E(X^2) - \mu^2$

$Var(X) = E[(X-\mu)^2]$

$= E[X^2 - 2X\mu + \mu^2]$

$= \sum_x (x^2 - 2x\mu + \mu^2)P(X = x)$

$= [\sum_x x^2 P(X = x)] - [2\mu \sum_x xP(X = x)] + \mu^2 \sum_x P(X = x)$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2$

Prove that $Var(aX + b) = a^2 VarX$

$$Var(aX + b) = E[(aX + b)^2] - [E(aX + b)]^2$$

$$= E[a^2X^2 + 2abX + b^2] - [aE(X) + b]^2$$

$$= a^2E(X^2) + 2abE(X) + b^2 - [a^2[E(X)]^2 + 2abE(X) + b^2]$$

$$= a^2E(X^2) - a^2[E(X)]^2$$

$$= a^2VarX$$

The random variables X_i , for $i = 1$ to 100, are independent, and $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$

Find:

(i) $Var(X_1)$

(ii) $Var(X_1 + X_2 + \cdots + X_{100})$

(iii) $Var(100X_1)$

(iv) $Var(X_1 - X_2)$

Solution

(i) $Var(X_1) = E\left({X_1}^2\right) - (E(X_1))^2$

$= \left\{\frac{1}{2}.1 + \frac{1}{2}.1\right\} - 0 = 1$

(ii) $Var(X_1 + X_2 + \cdots + X_{100})$

$= Var(X_1) + Var(X_2) + \cdots + Var(X_{100})$

(as the X_i are independent)

$= 100(1) = 100$

(iii) $Var(100X_1) = 100^2 Var(X_1) = 10000$

(iv) $Var(X_1 - X_2) = 1^2 Var(X_1) + (-1)^2 Var(X_2)$

$= 1 + 1 = 2$

Where X & Y are not necessarily independent:

$Var(aX \pm bY) = a^2VarX + b^2VarY \pm 2abCov(X,Y)$,

where $Cov(X,Y) = E(XY) - E(X)E(Y)$