STEP - Differential Equations

Solve $sinx\frac{dy}{dx} + secx.y = cosx$

Solution

Before we find an integrating factor, can we rearrange the LHS into the form $P(x)\frac{dy}{dx}+P'(x)y$?

Dividing through by $cosx$ gives

$tanx\frac{dy}{dx}+sec^2x.y=1$,

so that $\frac{d}{dx}(ytanx)=1$

and hence $ytanx=x+C$,

so that $y=(x+C)cotx$

Note: Finding the IF here is quite time-consuming:

First of all, $\frac{dy}{dx}+\frac{1}{cosxsinx}.y=cotx$

Then IF = $\exp\{\int\frac{1}{cosxsinx}dx\}$

$I=\int\frac{1}{cosxsinx}dx=2\int\frac{1}{sin2x}dx$

$=2\int\frac{sin2x}{sin^2 2x}dx=2\int\frac{sin2x}{1-cos^2 2x}dx$

Let $u=cos2x$, so that $du=-2sin2x\ dx$

and $I=-\int\frac{1}{1-u^2}\ du=-\frac{1}{2}\int\frac{1}{1-u}+\frac{1}{1+u}\ du$

$=-\frac{1}{2}\{-\ln|1-u|+\ln|1+u|\}$

$=\frac{1}{2}\ln\left|\frac{1-u}{1+u}\right|=\frac{1}{2}\ln\ |\frac{1-cos2x}{1+cos2x}|$

$=\frac{1}{2}\ln\left|\frac{2sin^2x}{2cos^2x}\right|=\frac{1}{2}\ln(tan^2x)=\ln\ |tanx|$

So $IF=\exp\{\ln|tanx|\}=tanx$

Solve $\frac{dy}{dx} = x + y$ by:

(a) finding an integrating factor

(b) making the substitution $z = x + y$

Solution

(a) $\frac{dy}{dx} = x + y \Rightarrow \frac{dy}{dx} - y = x$

I.F. $= \exp\{\int -1\, dx\} = e^{-x}$

Then $e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{-x}$

$\Rightarrow \frac{d}{dx}(ye^{-x}) = xe^{-x}$

$\Rightarrow ye^{-x} = \int xe^{-x}\, dx = x(-e^{-x}) - \int -e^{-x}\, dx = -xe^{-x} - e^{-x} + C$

$\Rightarrow y = Ce^{x} - 1 - x$

(b) $\frac{dy}{dx} = x + y \Rightarrow \frac{d}{dx}(z - x) = z$

$\Rightarrow \frac{dz}{dx} - 1 = z$

$\Rightarrow \frac{dz}{dx} = z + 1$

$\Rightarrow \int \frac{1}{z+1}\, dz = \int dx$

$\Rightarrow \ln|z + 1| = x - lnC$

$\Rightarrow C(z + 1) = e^{x}$

$\Rightarrow y = z - x = Ae^{x} - 1 - x$

In general, for $\frac{dy}{dx} = f(x + y)$, let $z = x + y$,

so that $\frac{dy}{dx} = \frac{dz}{dx} - 1$

and $\frac{dz}{dx} - 1 = f(z)$ etc

To convert $x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=0$ to $\frac{d^2y}{du^2}+c\frac{dy}{du}+dy=0$ (*)

Which of the following substitutions works: $u=e^x$ or $x=e^u$?

[To convert $x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=0$ to $\frac{d^2y}{du^2}+c\frac{dy}{du}+dy=0$ (*)

Which of the following substitutions works: $u=e^x$ or $x=e^u$?]

Solution

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

Now, $u=e^x\Rightarrow\frac{du}{dx}=u$,

and $x=e^u\Rightarrow\frac{du}{dx}=\frac{1}{\left(\frac{dx}{du}\right)}=\frac{1}{x}$

In the latter case, $\frac{dy}{dx}=\frac{dy}{du}(\frac{1}{x})$, and $x\frac{dy}{dx}=\frac{dy}{du}$

Then $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{du}\left(\frac{1}{x}\right)\right)=\left(\frac{d^2y}{du^2}.\frac{du}{dx}\right)\left(\frac{1}{x}\right)+\frac{dy}{du}(-\frac{1}{x^2})$

$=\frac{1}{x^2}(\frac{d^2y}{du^2}-\frac{dy}{du})$

So $x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=0$ becomes

$\left(\frac{d^2y}{du^2}-\frac{dy}{du}\right)+a\frac{dy}{du}+by=0$

ie $\frac{d^2y}{du^2}+(a-1)\frac{dy}{du}+by=0$

Show that $\frac{dy}{dx} = f(\frac{y}{x})$ can potentially be solved by making a substitution.

Solution

Let $z = \frac{y}{x}$, so that $y = xz$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$

So $\frac{dy}{dx} = f(\frac{y}{x})$ becomes $z + x\frac{dz}{dx} = f(z)$

and $\int \frac{1}{f(z)-z} dz = \int \frac{1}{x}\ dx$

Solve $\frac{dy}{dx} = \frac{x^3+4y^3}{3xy^2}, x > 0$

[Solve $\frac{dy}{dx} = \frac{x^3+4y^3}{3xy^2}$, $x > 0$]

Solution

Let $z = \frac{y}{x}$, so that $\frac{dy}{dx} = z + x\frac{dz}{dx}$, as previously.

Then $z + x\frac{dz}{dx} = \frac{1}{3z^2} + \frac{4z}{3}$

and $x\frac{dz}{dx} = \frac{1}{3z^2} + \frac{z}{3}$

so that $3\int \frac{1}{\frac{1}{z^2}+z}\ dz = \int \frac{1}{x} dx$

and $lnx = \int \frac{3z^2}{1+z^3} dz = \ln(1+z^3) + lnC$

$\Rightarrow x = C(1+z^3)\quad [C > 0]$

$\Rightarrow (\frac{y}{x})^3 = Ax - 1\ \ [A = \frac{1}{C}]$

$\Rightarrow y^3 = (Ax-1)x^3$